Bertrand-Edgeworth Competition Under Uncertainty*

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Abstract

This paper considers a Bertrand Edgeworth Duopoly where rivals cost are private information. It brings together the full information Bertrand Edgeworth model of Kovenock-Deneckere and the Bertrand under uncertainty model of Spulber. Firms set prices in a game where demand and capacities are common knowledge. The setting corresponds to a first price auction with capacity constraints. An increasing pure strategy Bayes-Nash equilibrium always exists. The equilibrium is not necessarily continuous. Firms charge supracompetitive prices. Prices for a given cost can be lower than in the simple Bertrand under uncertainty model.

Keywords: Bertrand-Edgworth Competition, Oligopoly Pricing, Capacity Constraints, Cost Uncertainty, First-Price Auctions

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1 Introduction

Since Cournot's "Researches into the Mathematical Principles of Wealth" in 1838 economists examined various forms of short run competition. Even though all these models try to explain firms' real-world behavior, the results differ fundamentally. Cournot's textbook model is easy and convincing. Firms set quantities and earn positive profits in equilibrium. Profits decrease in the number of firms. These results seem quite reasonable. In its simplicity the model is an instructive approximation for a more complex process of profit building.

But the model has some serious drawbacks. In his fundamental critique in 1883, Joseph Bertrand claims that firms set prices. While this objection is important, it is hard to verify whether firms set quantities or prices. Beside this dogmatic exception there is a theoretical hitch. Price determination in the Cournot model implies the adverse technical side effect of an auctioneer. Firms inform the auctioneer about the quantities they wish to produce. The auctioneer computes the market clearing price. Firms sell exactly their quoted quantities. The auctioneer rules out any additional trade.

The auctioneer is widespread in the literature but quite suspicious. A lot of theorists argue that the auctioneer is still an accurate metaphor for price building in free markets - an institution as the invisible hand. But the Bertrand model shows that if firms set prices, we do not need the auctioneer anymore. From an economic point of view the Bertrand model is as simple as the Cournot model. Firms set prices given market demand. The market allocates demand to the firm with the lowest price. Finally, firms produce the quantity demanded.

The Bertrand model is not uncontroversial. In equilibrium firms typically make zero profits - something we do not observe in reality. Additionally, the analysis is much more complex than in the Cournot model. Because of discontinuities the existence of an equilibrium in a Bertrand game is not trivial. Furthermore, uniqueness of equilibrium requires demanding assumptions.

Basic Cournot and Bertrand competition are quite simplistic. Edgeworth and Stackelberg showed that the outcome depends on the strategic variable and on timing. Later on, economists added new features of short run competition in order to make the model more realistic. Capacity constraints (Edgeworth 1925), Product Differentiation (Hotelling 1929) and repeated games (Friedman 1971) induced equilibria in which firms make positive profits. In 1983, David Kreps and José Scheinkman connected the story of Bertrand with the result of Cournot. In a two period model they show that under certain circumstances the Cournot outcome may occur if firms set prices in a capacity constrained game. In the first period firms build up capacities at possibly zero costs. In the second period firms produce at zero marginal costs up to their capacity level. Production beyond the capacity constraint is infinitely costly. It turns out that the Cournot outcome is the unique subgame perfect equilibrium. The model by Kreps and Scheinkman was a breakthrough. Capacity precommitment established the basis for a new branch in the literature. The convincing idea that firms compete in quantities in the long run but set prices in the short run is widely accepted today.

While allowing firms to set prices, the unloved auctioneer drops out of the scene. However, there are still a lot of simplistic assumptions. Dan Kovenock and Ray Deneckere showed that when unit costs differ, the outcome is no more necessarily Cournot. The firm with lower costs has an incentive to price its opponent out of the market.

The Bertrand-Edegworth outcome crucially depends on residual demand the firm with the higher price faces. Kreps and Scheinkman assume the efficient rationing rule. Davidson and Deneckere (1986) show that with different rationing rules the result may change.

But capacity precommitment is not the only possibility to prevent price setting firms from a zero profit outcome. An important source is uncertainty. If price setting firms face cost uncertainty, aggregate profits increase (see Spulber 1995). But the result of Bertrand competition under cost uncertainty is not completely innocuous. While adding asymmetric information makes the model more realistic, the result is not completely satisfying. In equilibrium almost every firm makes positive expected profits (except a bad luck firm drawing the highest possible costs). But ex post only the firm with the lowest price, i.e. the firm with the lowest costs, makes positive profits. Every other firm serves zero demand and earns zero profits. Again, this is not what we observe in reality. Typically, in oligopoly markets several firms compete, each making positive profits.

In this paper we try to solve the problem of only one firm with positive profit. To do so we add cost uncertainty to Bertrand Edgeworth competition. In fact, we connect the Bertrand Edgeworth model with the Bertrand under Uncertainty model. We reexamine Bertrand-Edgeworth competition with the assumption that unit costs are private information. Firms cannot observe the competitor's marginal costs but know the cost distribution.

This model is important for several reasons. On the one hand, analysis of Bertrand Edgeworth under cost uncertainty is important from an Industrial Organization point of view. Bertrand Edgeworth competition is an appropriate description for some branches where capacities are typically finite and competition takes place in prices. The most important branch is the electricity market. A lot of papers about electricity markets have been published recently.

On the other hand, the analysis is important for auction theory. As a matter of fact, the setup is similar to a variable quantity auction with capacity constraints. The correspondence in analysis between Bertrand competition an auction theory is an achievement of Old Age. A variable quantity auction where firms have capacity constraints exists in real life. Assume a government has a fixed budget for road maintenance. Since governments often run a balanced budget, quantity will depend on the price the winning firm charges. Furthermore, it is possible that a small construction company is unable to engineer the whole street because of capacity constraints. The existence of capacity constraints changes equilibrium pricing.

It is beyond the scope of this paper to provide a full theory of Bertrand Edgeworth under uncertainty. We locate some basic phenomena like discontinuous equilibria or the fact that capacity constraints may lower equilibrium prices.

We organize the paper as follows. Section 2 presents the basic model. Sections 3 and 4 discuss preliminaries. Results of Cournot Competition under Uncertainty (CU) and Bertrand Competition under uncertainty (BU) are necessary to understand the nature of short run competition under uncertainty. Section 5 deals with results and properties of the equilibrium pricing in a low bid auction. In Section 6 we introduce Bertrand Edgeworth competition under uncertainty. Then, Section 7 extends the theory to discontinuous equilibria. Last, Section 8 concludes the paper.

In the sequel of this paper we reexamine the capacity pre-commitment case with uncertainty.

2 The Model

We consider a duopolistic market in which firms produce a homogenous product. The market demand function is $\max(D(p), 0)$. We assume $D'(p) < 0, D''(p) \leq 0$. Demand is positive and twice continuously differentiable on the compact set $\mathcal{D} = \{c | c \in (0, \hat{p})\}$. D(0) is finite and $D(\hat{p}) = 0$. The corresponding inverse demand function $P(q_1, q_2)$ satisfies $P'(Q) < 0, P''(Q) \leq 0$.

Both firms have exogenous capacity constraints $k_i \in \mathcal{K}_i = R^+$, $i = 1, 2$. Capacity constraints are common knowledge. Each firm i produces the good at a constant unit cost c_i. Fixed costs are zero for both firms. The unit cost is drawn independently from a distribution with atomless cumulative distribution function $F(c_i) = F(c_j) = F(c)$ with support $\mathcal{C} = \{c | c \in (0, \hat{p})\}$. We choose this support for simplicity. But we can generalize this in a natural way. Marginal costs are private information. Each firm knows its own production cost but only the other firm's cost distribution. Since firms may be capacity constrained, we have to specify a particular rationing rule. As can easily be seen, different rationing rules yield different equilibria. In accordance with the literature we choose the efficient rationing rule. The problems related to this rationing rule have been widely discussed in the literature[1]. The efficient rationing rule has a natural interpretation. Consider an economy with identical consumers. The firm charging the lower price knows that there will be residual demand for the opponent. To improve the relative market position, the underselling firm may be interested in leaving the smallest demand for a given price to the other firm.[2] That is why it limits the quantity each consumer can buy to $k_i/D(p_i)$. In case of efficient rationing, the residual demand of the undersold firm i is obtained by shifting the market demand k_j units to the left. Efficient rationing is the worst case scenario for the undersold firm. However, the rule guarantees that the quantity sold from the firm with the lower price is allocated to the consumers with the highest valuation. Consumer surplus is therefore maximized and gains from trade are exhausted.

Firms compete in prices. They simultaneously set prices chosen from the set of feasible prices $\mathcal{P} = [c, \hat{p}]$[3]. With the efficient rationing rule firm's expected profits are:

$$\begin{aligned} E\pi_i\left(p_i\left(c_i\right), c_i\right) &= \left(p_i - c_i\right)\min\left(D_i\left(p_i\right), k_i\right)\Pr\left(p_i < p_j\right) \\ &\quad + \left(p_i - c_i\right)\min\left(D_i\left(p_i\right), k_i\right)\Pr\left(p_i = p_j\right)\Pr\left(c_i < c_j\right) \\ &\quad + \left(p_i - c_i\right)\min\left(\max\left(D\left(p_i\right) - k_j, 0\right), k_i\right)\Pr\left(p_i > p_j\right) \end{aligned}$$

In case of a tie we allocate the demand to the firm with lower marginal costs,

[1] For a discussion see Davidson-Deneckere (1986).

[2] This is a dynamic argument in a static game, but the relative market position may be important for funding reasons.

[3] We do not consider cases where firms set prices below unit costs since in a discrete approximation these strategies are strictly dominated. If strategy spaces are not compact we typically have a continuum of equilibria for Bertrand games. We allow for weakly dominated strategies in order to maintain the Bertrand knife edge result.

independent of capacity.[4]

This is a static game under incomplete information. We assume that firms act as Bayes-Nash players. A strategy is a function

$$p_i : \mathcal{C}_i \times \mathcal{K}_i \times \mathcal{K}_j \longrightarrow \mathcal{P}_i$$

That is a mapping from the type space $\mathcal{C}_i \times \mathcal{K}_i \times \mathcal{K}_j$ onto the space of feasible actions $\mathcal{P}_i$. Since capacities are exogenous and common knowledge, we simplify notation. In the remainder of this paper we simply write $p_i^*(c_i)$ for the equilibrium pricing scheme. A Bayes-Nash equilibrium (BNE) is a price vector $p^*(c)$ that specifies for both firms best answer given the strategy of the competitor.

In order to get close to uncertainty games we first investigate text book Cournot competition under cost uncertainty and Bertrand competition under uncertainty.

3 Cournot Competition Under Cost Uncertainty

Consider a Cournot duopoly where the inverse demand function is $P(Q) = P(q^1, q^2)$. Firms produce with marginal cost drawn independently from an identical distribution with cumulative distribution function $F(c)$ and support $\mathcal{C}$. Costs are private information but the distribution is common knowledge. A strategy is a mapping

$$q_i : \mathcal{C}_i \longrightarrow \mathcal{Q}_i$$

Let $R_i(q_j|c_i)$ be firm i's reaction function given its costs c_i. The reaction function is defined as:

$$R_i(q_j(c_j)|c_i) \equiv \arg\max_{q_i} \int_0^{\hat{p}} q_i \left[P(q_i + q_j(c_j)) - c_i\right] dF(c_j)$$

Because of our concavity assumption the reaction functions are downward sloping and a stable equilibrium exists. For a general demand function the equilibrium is difficult to characterize. But when demand is linear the equilibrium

[4] It turns out that the breaking rule does not matter in Bayesian Nash equilibrium for strictly increasing equilibrium pricing schemes. A tie takes place with probability 0.

is easy to derive. There is a unique BNE of the form $q_i = a_i + b_i c_i$[5]. We call the solution to this problem $q_i^{cu}(c_i)$. We compute the equilibrium in the simplest case with a uniform cost distribution:

$$D(p) = \max(1-p, 0) \qquad c \sim U(0,1)$$

The equilibrium is

$$q_i^{cu}(c_i) = \frac{5}{12} - \frac{c_i}{2} \quad \text{for} \quad c \in \left(0, \frac{2}{3}\right)$$

Each firm behaves like it would play against a firm of an expected cost type. [6] The equilibrium price will be $P = \frac{1}{6} + c_1 + c_2$.

4 Bertrand Competition Under Cost Uncertainty

The main source for Bertrand competition under cost uncertainty (BU) is Spulber (1995). The setup is similar to our model. Marginal cost is private information. But differently, BU implies "the winner takes it all" competition. The low cost firm serves the entire market. Its opponent has zero demand. To solve this problem we rely on results from auction theory. It turns out that this problem is identical to a variable quantity auction (See Maskin and Riley 1984). Firms act as Bayes Nash Players. Expected profits are equal to

$$\pi_i(p_i(c_i), c_i) = (p_i(c_i) - c_i) D(p_i(c_i)) G(p_i(c_i))$$

where $G(p_i(c_i))$ denotes the probability that firm i sets the lower price. A strategy is a mapping

$$p_i : \mathcal{C}_i \longrightarrow \mathcal{P}_i$$

We solve for a symmetric increasing equilibrium.

[5] See Basar (1978).

[6] Please note that this is not true in general but for the linear example.

Theorem 1 (Spulber 1995): In the Bertrand under Uncertainty model firms set prices according to the differential equation, initial condition and the side constraint:

$$p^{*\prime}(c) = \frac{f(c)}{1-F(c)} \frac{\pi(p^*(c), c)}{\frac{\partial \pi(p^*(c),c)}{\partial p^*(c)}}$$

$$\pi(p(\hat{p}), \hat{p}) = 0$$

$$\pi(p^*(c), c) \geqq 0$$

We write the differential equation in a compact form

$$p^{*\prime}(c) = v(c) \frac{\pi(p^*(c), c)}{\pi_p(p^*(c), c)}$$

where $v(c)$ is the hazard rate.

Proof of Theorem 1 See Spulber 1995

We call the solution of this problem p^{BU}. Of course, Bertrand competition is always a special case of Bertrand Edgeworth competition, independent of uncertainty. In section 6 we show that high cost firms may set the same prices as in the Bertrand under uncertainty model.

The problem with a Bertrand Duopoly under uncertainty is that in equilibrium every firm (except the firm drawing costs of $\hat{p}$) has ex ante positive profits. Ex post, however, only one firm generates positive profits, since all other firms face no demand. In reality we do not observe this knife-edge result. We do not observe a monopoly in each market. With capacity constraints, we get rid of this. That is one reason why Bertrand Edgeworth competition entered text books in the early 20th century: Its predictions often coincide with real world practice.

5 Variable Quantity Auction with Capacity Constraints

To understand the nature of Bertrand-Edgeworth Competition with capacity constraints we start with a simple example of a low bid auction. Two firms have exogenous capacity constraints $k_i = k_j = k$. Demand is D(p)=max$(1 - p, 0)$. Marginal costs are private information and independently drawn from a (0,1) uniform distribution. Firms simultaneously set prices. The firm with the lower price serves the market up to its capacity. The overbidder cannot serve residual demand and has therefore zero profit. This, at first glance, is not very realistic since rationing occurs at the market level. But the idea that the winner of an auction can buy as much as he wants/can is reasonable. Occasionally, in procurement auction only one seller delivers its capacity though demand would be higher.

Firm i's expected profit is:

$$E\pi_i\left(p_i\left(c_i\right), c_i, k\right) = \left(p_i\left(c_i\right) - c_i\right) \min\left(1 - p_i\left(c_i\right), k\right)\left(1 - c_i\right).$$

Despite the discontinuity effect, the problem is identical to the BU model.

$$E\pi_i\left(p_i\left(c_i\right), c_i, k\right) = \pi_i\left(p_i\left(c_i\right), c_i\right) G(c_i) = \pi_i\left(p_i\left(c_i\right), c_i\right)\left(1 - F\left(c_i\right)\right).$$

The fundamental differential equation from section 4 applies here too:

$$\frac{\Delta p^*\left(c\right)}{\Delta c} = v\left(c\right)\frac{\pi\left(p^*\left(c\right), c\right)}{\pi_p\left(p^*\left(c\right), c\right)}.$$

The simple difference to the standard BEU case is that we have, depending on the price, two different cases. The winning firm is either capacity constrained or not. Assume $\exists\ c_i$ s.t. $p_i^*\left(c_i\right) > 1 - k$. In this case the underbidding firm is not capacity constrained and we are in the BU case. The solution has to satisfy the differential equation:

$$p^{*\prime}(c) = \frac{(p(c) - c)(1 - p(c))}{(1 - 2p(c) + c)(1 - c)},$$

the initial condition and the side constraint:

$$p(1) = 1 \qquad \pi(c) \geq 0.$$

The solution to this differential equation is:

$$p^u(c) = \frac{1}{3} + \frac{2}{3}c.$$

Assume there exists a c_i s.t. $p_i^*(c_i) \leq 1 - k$. In this case the firm is capacity constrained and we have $\pi(p^*(c), c) = (p^*(c) - c)k$. The solution has to satisfy the differential equation:

$$p'(c) = \frac{\pi}{\pi_p(1 - c)} = \frac{(p(c) - c)k}{k(1 - c)} = \frac{p(c) - c}{(1 - c)}$$

Now we have to find the right initial condition. Define $\tilde{c}_i$ such that $p_i^*(\tilde{c}_i) = 1 - k$. At this $\tilde{c}$ firms' expected profit in the constrained case must be equal to the profit in the unconstrained case. This implies that the equilibrium pricing scheme will be continuous on the set $\mathcal{D}$. So the solution to the last differential equation must cross the solution in the BU case exactly at a cost $\widetilde{c}$, where the price is $1 - k$.

$$p^u(\widetilde{c}) = \frac{1}{3} + \frac{2}{3}\widetilde{c} = 1 - k$$

We can solve for $\tilde{c}$ and obtain $\tilde{c} = 1 - \frac{3}{2}k$. The initial condition and the side constraint are

$$p\left(1 - \frac{3}{2}k\right) = 1 - k \qquad \pi(c) \geq 0$$

The solution to this differential equation is:

$$p(c) = \frac{4 - 4c^2 - 3k^2}{8(1 - c)}.$$

Putting the things together we have the following equilibrium:

$$p^*(c) = \begin{cases} \frac{4-4c^2-3k^2}{8(1-c)} & \text{if} \quad 0 \le c \le 1 - \frac{3}{2}k \\ \frac{1}{3} + \frac{2}{3}c & \text{if} \quad 1 - \frac{3}{2}k < c < 1 \\ c & \text{if} \quad 1 \le c \end{cases}$$

In the capacity constrained region, the solution crucially depends on the hazard rate of the distribution function F. See figure 1 for the equilibrium pricing scheme and figure 2 for the profit function for k=0.2.

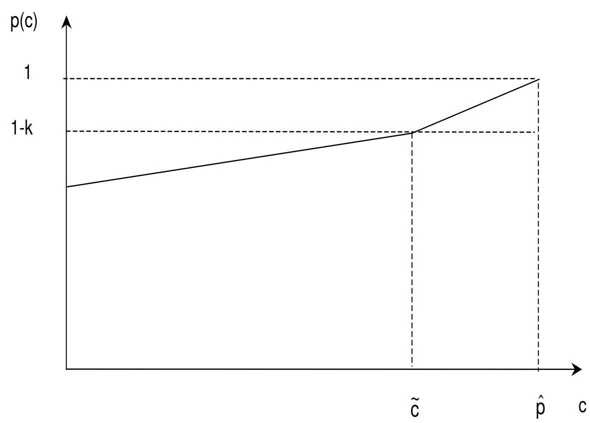

Figure 1: Bayes Nash Equilibrium in the Low Bid Auction

Proposition 1 : In a symmetric Bayes Nash Equilibrium of the variable quantity low bid auction with equal capacity constraints the pricing scheme $p_i^*(c_i, k)$ satisfies the following properties:

1. $p_i^*(c_i, k)$ is continuous.
2. $p_i^*(c_i, k)$is increasing in c_i.
3. $p_i^*(c_i, k)$ is piecewise differentiable on the ranges $[0, \widetilde{c}]$, $[\widetilde{c}, \hat{p}]$.
4. $p_i^*(c_i, k)$ is piecewise convex in c on the ranges $[0, \widetilde{c}]$ and $[\widetilde{c}, \hat{p}]$.

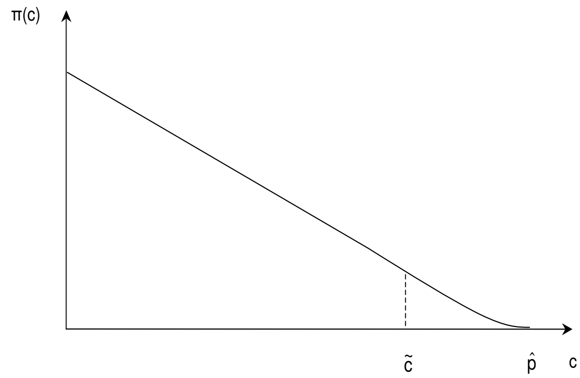

Figure 2: Expected Profits in the Low Bid Auction

5. For any $k > 0$, $p_i^*(c_i, k)$ is strictly lower than the monopoly price.

Proof of Proposition 1 :

1. Assume the equilibrium would not be continuous. The point of non-continuity has to be $\tilde{c}$. This would imply that any price between the two branches would be dominated (as the probability would not change if one firm would play this price). But the profit function in the constrained case is strictly increasing. So we have a contradiction.

2-4 are straightforward. In the capacity constrained region, the properties are easy to check. For the BU region see Spulber (1995). Number 5 is not intuitive. Suppose $p^*(c) = p^m(c)$. At the monopoly price we know $\frac{\partial \Pi}{\partial p} = 0$. Hence, the first order condition in the capacity constrained range would become $p\prime(c) \rightarrow \infty$. This is possible if and only if either $k = 0$ or $c = \hat{p}$ But $\Pi(\hat{p}) = 0 \neq \Pi^m$. This completes the contradiction. ■

Proposition 2 : In the symmetric Bayes Nash Equilibrium the profit function $\Pi(p_i^*(c_i, k)$ satisfies the following properties:

1. $\Pi_i\left(p_i(c_i), k)\right)$ is decreasing in c_i, and concave in k.
2. $\Pi_i(p_i(c_i), k)$ is continuous.

3. $\Pi_i(p_i(c_i), k)$ is piecewise differentiable on the ranges $[0, \widetilde{c}], [\widetilde{c}, \hat{p}]$

4. $\Pi_i(p_i(c_i), k)$ is strictly lower than the monopoly profit for any $k > 0$.

Proof of Proposition 2 : All these properties except one follow directly from the properties of the equilibrium pricing scheme. As in equilibrium the winning-probability is independent of price and capacity, concavity in capacity is obvious.[7]

6 Bertrand Edgeworth Under Uncertainty

6.1 General Results

The analysis is similar to the simple low bid auction. We use techniques from auction theory. The only difference is that the overbidder has the possibility to serve residual demand.

Under full information the game $G(k_1, k_2, c_1, c_2)$ is discontinuous in p. But the existence of a mixed strategy equilibrium is guaranteed by Dasgupta and Maskin (1986). With uncertainty the analysis is easier. The discontinuity vanishes.

Lemma 1 : The game has a pure strategy Bayes Nash Equilibrium $p_i(c_i, k_i, k_j)$.

Proof of Lemma 1 : The payoff function

$$\begin{aligned}\pi_i\left(p_i\left(c_i\right), c_i\right) &= \left(p_i - c_i\right)\min\left(D_i\left(p_i\right), k_i\right)\Pr\left(p_i < p_j\right)\\ &\quad + \left(p_i - c_i\right)\min\left(D_i\left(p_i\right), k_i\right)\Pr\left(p_i = p_j\right)\Pr\left(c_i < c_j\right)\\ &\quad + \left(p_i - c_i\right)\min\left(\max\left(D\left(p_i\right) - k_j, 0\right), k_i\right)\Pr\left(p_i > p_j\right)\end{aligned}$$

is continuous in p because in the increasing equilibrium the winning probability is independent of p. We have conditional independence, atomless distributions for types and compact action spaces. All conditions of Milgrom and Weber (1985)

[7] Concavity is not intuitive. But as we will see in the sequel that this property makes life easier in the capacity pre-commitment game.

are fulfilled and the game has a pure strategy Bayes Nash Equilibrium. We may also apply theorem 2 of Dasgupta and Maskin (1986). ■

Next, we establish that the pricing scheme is non decreasing.

Lemma 2 : The pricing scheme $p_i(c_i|k)$ is nondecreasing in c_i.

Proof of Lemma 2 : Let w.l.o.g. $c_1 \geq c_2$. Let $p_i(c_i), i = 1, 2$ be the equilibrium price. Define $\psi^L(p_i)$ as the demand for firm i if it sets the lower price. Define $\psi^H(p_i)$ analogously. Assume $\exists\ c_1, c_2\ s.t.\ p_2(c_2) > p_1(c_1)$.
Optimality implies

$$E\pi\left(p_1|c_1\right) > E\pi\left(p_2|c_1\right)$$

$$E\pi\left(p_2|c_2\right) > E\pi\left(p_1|c_2\right)$$

We add the equations

$$\begin{aligned} & G_1\left(p_1 - c_1\right)\psi^L\left(p_1\right) + \left(1 - G_1\right)\left(p_1 - c_1\right)\psi^H\left(p_1\right) \\ > \ & G_2\left(p_2 - c_1\right)\psi^L\left(p_2\right) + \left(1 - G_2\right)\left(p_2 - c_1\right)\psi^H\left(p_2\right). \end{aligned}$$

Simplification yields

$$G_2\psi^L\left(p_2\right) + \left(1 - G_2\right)\psi^H\left(p_2\right) > G_1\psi^L\left(p_1\right) + \left(1 - G_1\right)\psi^H\left(p_1\right).$$

But expected demand is non-increasing in p. This completes the contradiction.
■

In the remainder of this paper, we want to be as simple as possible. We assume that $D(p) \equiv \max(1-p, 0)$. Depending on capacity level we can define 4 potential different cost regions. Define region $A_i \equiv \{c_i \in (0, \hat{p}) : p_i^*(c_i) \leq 1 - k_i - k_j\}$ as firm $i's$ cost region, where an equilibrium price, if it exists, is not bigger than the capacity clearing price $P(k_i + k_j) = 1 - k_i - k_j$.

Theorem 2 : If firm i's marginal cost lie within region A_i, this firm sets the capacity clearing price $p_i(c_i) = P(k_i + k_j) = 1 - k_i - k_j$. In other words, an equilibrium price is not lower than the market clearing price.

Proof of Theorem 2 : The highest price in this region is the market clearing price $P(k_i+k_j)$. At the market clearing price both firms are capacity constrained. Both sell their entire capacity. By quoting a price below the market clearing price, firm i cannot increase the demand for its good. But the markup per unit decreases. Hence, profits decrease. If a firm deviates upward, we are not in region A_i. ■

If capacities are large it may be that the capacity constraint does not bind. If so, firms set their prices independent of capacities. Then, the price in the BEU case is equal to the price in the BU case: $p_i(c_i, k_i, k_j) = p_i(c_i)$. Let $D_i \equiv \{c_i \in (0, \hat{p}) : p_i^{BU}(c_i) > 1 - \min(k_i, k_j)\}$.

Theorem 3 : If a firm's marginal costs lie within region D_i, this firms sets a price $p_i(c_i)$ according to the differential equation, the initial condition and the side constraint:

$$p^{*\prime}(c) = v(c)\frac{\pi(p^*(c), c)}{\pi_p(p^*(c), c)}$$

$$\pi(p(\hat{p}), \hat{p}) = 0$$

$$\pi(p^*(c), c) \geqq 0$$

Proof of Theorem 3 : See section 4.

Complexity arises due to non-differentiability of the demand function and the fact that the equilibrium strategy is not necessarily strictly increasing. There are (possibly) two additional regions where price setting behavior is more complex than in regions A_i and D_i. Define region $B_i \equiv \{c_i \in (0, \hat{p}) : 1 - k_i - k_j < p_i^*(c_i) \leq 1 - \max(k_i, k_j)\}$ and region $C_i \equiv \{c_i \in (0, \hat{p}) : 1 - \max(k_i, k_j) < p_i^*(c_i) \leq 1 - \min(k_i, k_j)\}$. Define finally S^X as the supremum of the set X. The next figure shows how an equilibrium should look like in an idealized situation for $k_j > k_i$: Please note that depending on capacities, some of the sets A,B,C can be empty. For our linear example, region D is not empty.

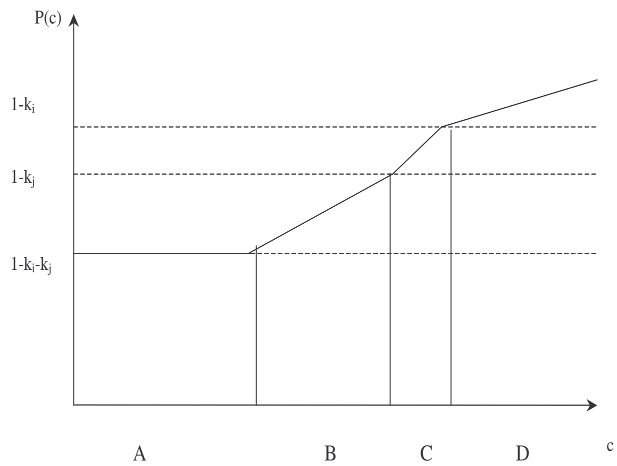

Figure 3: Idealized Equilibrium in BEU Competition

6.2 Symmetric Equilibria

In auction theory we typically assume that a symmetric equilibrium exists. Firms play strictly increasing strategies that are differentiable. This makes the analysis much simpler. But this may not be the case in this model. A continuously differentiable pricing scheme $p(c)$ does not exist in general. Analogous to the low bid auction we look for a symmetric, strictly increasing and continuous equilibrium. To be sure that a symmetric equilibrium exists we assume that both firms have the same capacities. As firms now are a priori identical at least one equilibrium should be a symmetric one. Of course this is a limiting assumption. But the change to asymmetric bidder models is more a question of technics than of economics. Since we prefer giving economic comprehension to solving systems of differential equations we restrict to the equal capacity model.

For any c, both firms should set the same price $p_i(c_i)$. Assume a continuous, symmetric equilibrium exists where both firms play strategies that are strictly increasing in c (on the sets B,C and D) . This implies that regions B and C coincide.

Theorem 4 : If a firm's marginal costs lie within region $B\bigcup C$ and capacities are such that the equilibrium is a continuous and symmetric one in which the equilibrium strategy is strictly increasing in c, firms set prices $p_i(c_i)$ among the differential equation

$$p'(c)=\frac{G'(c)\left(\pi^H-\pi^L\right)}{G(c)\frac{\partial\pi^L}{\partial p}+(1-G(c))\frac{\partial\pi^H}{\partial p}}$$

the initial condition

$$p\left(s^C\right)=1-k$$

and the side constraint

$$p_i(c_i)\geq c_i$$

Proof of Theorem 4 : Firm i's expected profit is equal to

$$\begin{aligned}E\pi_i(p_i(c_i),c_i) &= \pi^L(p(c),c)G(c)+\pi^H(p(c),c)(1-G(c))\\ &= (p(c)-c)\min(D(p_i),k_i)(1-F(c_i))\\ &\quad+(p(c)-c)\min(\max(D(p_i)-k_j,0),k_i)F(c_i)\end{aligned}$$

By the revelation principle, the BNE can be represented as a direct mechanism that satisfies incentive and individual rationality constraint. The expected profit of a type c firm acting as a type y firm is:

$$E\pi_i(y,c)=\pi^L(p(y),c)G(y)+\pi^H(p(y),c)(1-G(y))$$

By the envelope theorem:

$$\frac{\partial E\pi_i(c)}{\partial c}=\frac{\partial\pi^L}{\partial c}(p(c),c)G(c)+\frac{\pi^H}{c}(p(c),c)(1-G(c))$$

This implies that $\frac{\partial E\pi_i(c)}{\partial c}<0$ for $c<\hat{p}$. $y=c$ is a global maximum of $E\pi_i(y,c)$. Rearranging the terms yields:

$$p'(c)=\frac{G'(c)\left(\pi^H-\pi^L\right)}{G(c)\frac{\partial\pi^L}{\partial p}+(1-G(c))\frac{\partial\pi^H}{\partial p}}$$

■

As can easily be seen the Bertrand under uncertainty model is just a special case of Bertrand-Edgeworth under uncertainty. If we set in this formula $\Pi^H = 0$ we obtain the Bertrand under uncertainty result.

The solution to this differential equation is the final part of the equilibrium $p_i^*(c_i)$. We fully characterized the equilibrium. The serious problem we have is, that such an continuous equilibrium does not always exist. We will see this in section 7.

As mentioned above, the Bertrand-Edgeworth under uncertainty model is formally identical to a variable quantity auction where firms are capacity constrained. Therefore, it is intuitive that each bidder equalizes marginal gains from a higher price (through a higher profit in the case of winning and maybe in the case of losing) with the marginal costs (through a lower probability of winning and perhaps of a lower profit in the case of losing).

6.3 Example

We compute an equilibrium pricing scheme for the following case:

$$D(p) = \max(1-p, 0),$$

$$c \sim U(0,1),$$

$$k_i = k_j = k = 0.5.$$

Assume $\exists\ c$ s.t. $p_i^*(c_i) > 1-k$. In Section 4 we saw:

$$p_i^*(c_i) = \frac{1}{3} + \frac{2}{3}c_i$$

$$S^C = 0.25$$

Assume $\exists\ c$ s.t. $1-2k < p_i^*(c_i) < 1-k$. The solution has to satisfy the following differential equation:

$$p'(c) = \frac{(p(c)-c)(p(c))}{(1-c)0.5 + c(0.5 - 2p(c) + c)}$$

and the initial condition

$$p\,(0.25) = 0.5.$$

The solution to this differential equation is a mess but an analytical solution exists.

Assume finally $\exists\ c$ s.t. $p_i^*(c_i) \leq 1 - 2k$. This implies that both firms are capacity constrained. By Theorem 1:

$$p = 0.$$

In this example the first and the second case are relevant. Set A is empty and the sets B and C coincide. We do not show the algebraic solution. Figure 4 is a plot of the equilibrium pricing scheme.

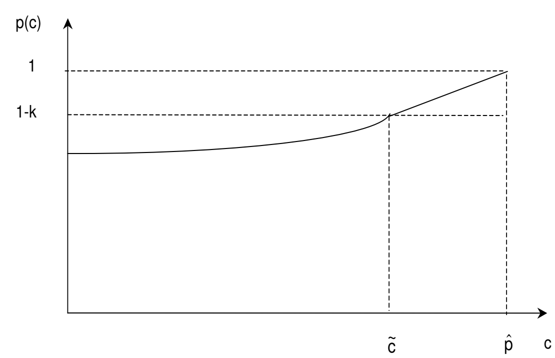

Figure 4: Symmetric Bayes Nash Equilibrium in the BEU game

6.4 Properties

Proposition 3 : In the unique continuous symmetric Bayes Nash Equilibrium the pricing scheme $p_i^*(c_i, k_i, k_j)$ satisfies the following properties:

1. p_i^* is constant on the set A and strictly increasing on the sets B,C,D.
2. p_i^* is piecewise differentiable on sets $[A, B, C, D]$.
3. p_i^* is convex on the set A and D.
4. On the set D p_i^* lies strictly below the monopoly price.
5. The price in the BEU case may lie below the price in the BU case.

Proof of Proposition 3 : We prove 1 in lemma 2. For region A differentiability is trivial. For region D Maskin and Riley show that the equilibrium bid function is unique, differentiable and strictly increasing. We may proof the properties the same way here. For regions B and C we compute the partial derivative. It turns out that under our assumptions the partial derivative is finite. By Lifschitz the solution is continuous and differentiable.
3. and 4. See Spulber (1995). It turns out that these properties do not necessarily carry over to the BEU case (i.e. regions B and C) .
5. Compare the fundamental differential equations for the BU case and the constrained case. For the constrained case we have

$$p'(c) = \frac{G'(c)\left(\pi^H - \pi^L\right)}{G(c)\frac{\partial \pi^L}{\partial p} + (1 - G(c))\frac{\partial \pi^H}{\partial p}}$$

We observe an additional term $(1 - G(c))\frac{\partial \pi^H}{\partial p}$ in the denominator. In equilibrium this term may be negative implying that the partial derivative is larger than in the BU case. For the same reason the BU properties 3 and 4 do not necessarily hold for regions B and C. For an example see section 6.3.

Because of non differentiability the profit function has two branches. An analytical solution for the left branch does not necessarily exist. The right branch is identical to the low bid auction.

Proposition 4 : The expected profit function of a firm i with costs c_i facing capacities k has the following properties

1. It is strictly positive on the range $[0, \hat{p})$.
2. It is strictly decreasing in c_i.
3. It is continuous
4. It is piecewise differentiable on the sets A,B,C and D respectively.

Proof of Proposition 4 :

1./2. See theorem 3. Property 4 follows directly from the properties of p^*. Point 3 is slightly more complicated. We will see in the next section that the pricing scheme is not necessarily continuous. The remaining question is whether at least the profit function is continuous. It is clear that our pricing scheme has at most one point of discontinuity at the supremum of the set C. Further, we know that for each type c both firms set exactly one price. By proposition 4 the profit function is continuous for both branches on the sets C and D. Suppose the equilibrium profit function would not be continuous at S^C. This implies that the profit on the right hand branch would be substantially lower than on the left hand side (because of the jump). This implies that by switching to a price above $1 - k$ profits decrease strongly. But if this discontinuity would exist, firms could do better. By staying in the constrained region C profits would decrease continuously. Since this solution still solves the fundamental differential equation we have in fact a solution of our problem. This completes the proof. ■

It is noteworthy that if firms have capacity constraints prices can be lower than in the BU case (for some costs). This is a fundamental difference to the full information case, where binding constraints yield higher (or not lower) prices in equilibrium. An economic reasoning behind may be that with a capacity constraint, there is also residual demand for the case where own costs are higher. If a firm serves residual demand, it has to set a low price in order to have positive demand. Lowering the price increases the probability of serving original demand. But there is an additional effect in the BEU model. A lower price can increase profits for the case when a firms serves residual demand even though it lowers the profit in the case of winning. The increase in (residual) demand can overcompensate the decrease in the price. These effects together yield a region where the BEU price is lower than the BU price.

As mentioned above, cost uncertainty is an important feature in Bertrand games. The effect of uncertainty on aggregate profits is not clear. While lowering the competition effect (compare the standard Bertrand paradox with the Bertrand under uncertainty game), firms act as risk averse bidders in the auction. There is a lot of space for future research. One theme may be the effect of information sharing coalitions in the sense of Shapiro (1986).

7 Discontinuities in the pricing scheme

If capacities are small an equilibrium is not necessarily continuous. In the fundamental differential equation the nominator is always positive. But the denominator can be negative:

$$p'(c) = \frac{G'(c)\left(\pi^H - \pi^L\right)}{G(c)\frac{\partial \pi^L}{\partial p} + (1 - G(c))\frac{\partial \pi^H}{\partial p}}$$

For our example the denominator is negative if

$$(1 - c)\,k + c\,(1 - k - 2p\,(c) + c) < 0$$

$$k - 2ck + c - 2cp\,(c) + c^2 < 0$$

$$k < \frac{2cp^*(c) - c + c^2}{1 - 2c}$$

If we fill in the former initial condition, the pricing scheme would be decreasing - a contradiction to lemma 2 (the denominator in the fundamental differential equation is negative). Hence

$$p\left(1 - \frac{3}{2}k\right) = 1 - k$$

is not the right initial condition. The differential equation we have to solve is unchanged. To find the new initial condition in regions B and C we need to assure that at the argument of the initial condition no firm has a higher profit by deviating from the pricing scheme proposed.

The equilibrium pricing scheme has still two branches. The discontinuity will arise at S^C. To avoid misunderstandings we rename this point as $S^C \equiv c^D$ (point of discontinuity). $p^l(c^D)$ as the right end of the lhs price branch. Let $p^r(c^D)$ be the left end of the rhs price branch respectively. Continuity of the profit function implies that the right endpoint of the left hand branch must give the same profit as the left endpoint $\pi(p^l(c^D) = \pi(p^r(c^D))$. Furthermore, any price in between these two endpoints must imply a lower expected profit (no deviating condition). Hence, the derivative of the expected profit with respect to p must be 0 at $p^l(c^D)$. In other words: If a firm with marginal costs c^D deviates and sets a price slightly above $p^l\left(c^D\right)$ the probability of being the underbidder does not change. This is true, since the other firm playing the discontinuous equilibrium pricing scheme does not play any price in between.

To recapitulate, the following conditions must hold at a point of discontinuity:

$$\pi\left(p^l\left(c^D\right), c^D\right) = \pi\left(p^h\left(c^D\right), c^D\right),$$

$p^l\left(c^D\right)$ must satisfy

$$p'\left(c\right) = \frac{G'\left(c\right)\left(1-2k-p\left(c\right)+c\right)}{G(c)k+\left(1-G\left(c\right)\right)\left(1-k-2p\left(c\right)+c\right)},$$

$$p^l = \arg\max_p \left(p-c^D\right)\left(1-p\right)G\left(p\left(c^D\right)\right)+\left(p-c^D\right)\left(1-k-p\right)\left(1-G\left(p\left(c^D\right)\right)\right),$$

and $p^h\left(c^D\right)$ must satisfy

$$p^h = \arg\max_p \left(p-c^D\right)\left(1-p\right)G\left(c^D\right)$$

$$p'\left(c\right) = \frac{G'\left(c\right)\left(p\left(c\right)-c\right)\left(1-p\left(c\right)\right)}{G\left(c\right)\left(1-2p\left(c\right)+c\right)}$$

No closed form solution exists for the equilibrium. The next figure shows the equilibrium pricing scheme for $k = 0.2$.

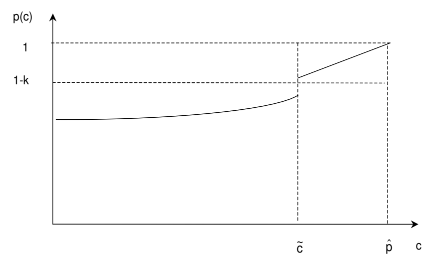

Figure 5: Discontinuity of p(c)

The discontinuity arises from a jump from the constrained profit function to the unconstrained profit function. For $p_i^* \leq 1 - k$ firm i is capacity constrained. The profit function takes the form $\pi_i(p_i(c_i), c_i) = (p_i(c_i) - c_i)(k_i)(1 - c_i) + (p_i - c_i)(1 - k_j - p_i)c$. As soon as $p_i^*(c_i) > 1 - k_i$ the capacity constraint for firm i is not binding and the residual demand for the overbidder is 0. The overbidder's profit function takes the form $\pi_i(p_i(c_i), c_i) = (p_i(c_i) - c_i)(1 - p_i)(1 - c_i)$. Since the ending point $p^l(c^D)$ of the unconstrained profit function coincides with a maximum, we jump from one peak to the other, with a discontinuity in the middle (see figure 6).

Why does this discontinuity not arise in the low bid auction? The reason is simple. At a point of discontinuity $\tilde{c}$ a constrained firm never has an incentive to lower its price from 1-k. The constrained profit function is strictly increasing. So the constrained firm will always prefer to stay on the constrained profit function to jump over to the unconstrained profit function. This is not the case in BEU where the constrained profit function is concave in p.

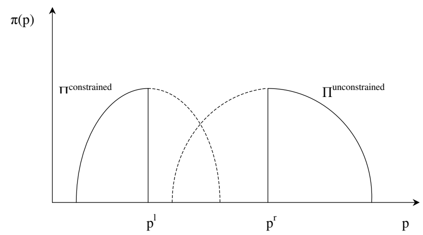

Figure 6: Discontinuity of p(c)

8 Conclusion

The present paper expands the Bertrand-Edgeworth model to cost uncertainty. This setup is identical to a variable quantity auction with capacity constraints.

The equilibrium pricing scheme can be separated into four regions. The equilibrium pricing function is not differentiable - it is not even necessarily continuous. If costs are low, prices may be independent of capacity level. For high costs and/or high capacities the model is identical to the Bertrand model with costs uncertainty. For medium costs, the equilibrium price can be lower than without capacity constraints. This is not intuitive at all but the effect of a residual demand can lower the price.

For some capacities equilibrium can be discontinuous even for a simple linear demand calibration. The discontinuity in the pricing scheme is noteworthy but very complicated to proof in reality. If firms play BEU it is not possible to discriminate between prices that are not played because cost differed and prices that are not played because they are in the discontinuity region.

We have seen that in the incomplete information case equilibria are in pure strategies while in the full information case there may be regions where equilibria necessarily occur in mixed strategies. This is a striking difference.

It is hard to compare a pure strategy with a mixed strategy equilibrium. In particular it is not possible to detect whether prices are higher or lower under uncertainty. But we can state that results are quite similar with uncertainty. Especially the fact that firms charge supracompetitive prices is preserved under uncertainty. *Ex ante* (almost) every firm has a positive profit. Ex post firms make positive profits, if costs are not too high. So we can solve the serious problem of the Bertrand under uncertainty model. In our case it is possible that both firms make positive profits while in the Spulber duopoly model there is always one firm that faces zero demand.

Some economists say that people do not play mixed strategies. These purists should be pleased with the result that Bertrand Edgeworth competition under uncertainty replicates more or less the results of the full information model but does not need the construct of fully mixed strategies.

In the literature, Bertrand Edgeworth competition became the generally accepted description for the electricity market. From the oscillations in prices economists deduced that this may be due to fully mixed strategies or that fully mixed strategies are the right description for this phenomenon, respectively.

In reality we observe markets where firms definitely face capacity constraints but we do not observe price oscillations. Examples may be goods like cars or books. But if firms face Bertrand Edgeworth competition but prices do not move in the sense of mixed strategies, there must be a mistake in the setup.

Of course, the objection that the model is inappropriate because firms play repeated games is always possible. But this dynamic argument would prohibit any static analysis. But static analysis in oligopoly models is still accurate and accepted. The initial point of any analysis will always be the one-shot interaction. And finally there will always be insights by a one-shot setting. Not only because in repeated games anything goes and equilibrium analysis is of little value.

A second objection would be that firms do not set prices but quantities. This would put the cart before the horse. Most of economist would support the Bertrand paradigm that firms set prices in the short run.

The right answer to the question asked is that in these markets firms do no play fully mixed strategies. The setup is not wrong but incomplete. If they play Bertrand Edgeworth competition they must be either in the pure strategy region or there must be an other explanation. Of course they may be in the pure strategy region because of capacity pre-commitment (Kreps and Scheinkman) or because of large capacities. But our paper may also explain why firms behave like this. The answer may be cost uncertainty. We do not want to judge what is more realistic, capacity pre-commitment or cost uncertainty. Both setups give special insights and both hypothesis should be tested empirically.

But what about the description of the electricity market? Nevertheless, our analysis may even provide an accurate description for this market. Some authors argue that prices in electricity markets vary over time since firms play mixed strategies. This argument assumes that Bertrand Edgeworth competition is an approximation for a more complex (repeated) game. But it is also possible that these oscillation do no emerge because of mixed strategies but because of varying costs. Suppose costs vary over time and firms face one shot interactions every day. The result would be that prices vary over time.

References

[1] Basar, T., 1978, "Equilibrium Solutions in Static Decision Problems With Random Coefficients in the Quadratic Cost", IEEE Transactions on Automatic Control, AC-23(5), 960-962.

[2] Boccard, N., X. Wauthy, 2000, "Bertrand Competition and Cournot Outcomes: Further Results", Economics Letters, 68, 279-285.

[3] Budde, J.,R.F. Göx, 1999, "The Impact of Capacity Costs on Bidding Strategies in Procurement Auctions", Review of Accounting Studies, 4, 5-13.

[4] Chamberlin, E. H. 1933, "The Theory of Monopolistic Competition", Harvard Economic Studies, XXXVIII, Harvard University Press, Cambridge.

[5] Dasgupta, P., Maskin E., 1986, "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory", The Review of Economic Studies, 53, 1-26.

[6] Dasgupta, P., Maskin E., 1986, "The Existence of Equilibrium in Discontinuous Economic Games, II: Applications", The Review of Economic Studies, 53, 27-41.

[7] Davidson, C., Deneckere R., 1986, "Long-Run Competition in Capacity, Short-Run Competition in Price, and the Cournot Model", The Rand Journal of Economics , 17, 404-415.

[8] Deneckere, R., Koveneock, D., 1996, "Bertrand-Edgeworth duopoly with unit cost asymmetry", Economic Theory , 8, 1-25.

[9] Deneckere, R., Koveneock, D., 1992, "Price Leadership", The Review of Economic Studies , 59, 143-162.

[10] Edgeworth, F.Y. 1925, "The Theory of Pure Monopoly", in Papers Relating to Political Economy, 1, MacMillan,London.

[11] Fudenberg, D., J. Tirole, 1991, "Game Theory", MIT Press, London.

[12] Gal-or E. 1986,"Information Transmission–Cournot and Bertrand Equilibria", The Review of Economic Studies, 53(1), 85-92.

[13] Gibbons, R. 1992, "A Primer In Game Theory", FT Prentice Hall.

[14] Klemperer, P. D. 1999, "Auction theory: A guide to the literature", Journal of Economic Surveys, 13, 227-286.

[15] Kreps, D.,J. Scheinkman, 1983, "Quantity precommitment and Bertrand competition yield Cournot outcomes", The Bell Journal of Economics , 14, 326-337.

[16] Krishna, V., 2002, "Auction Theory" , Academic Press, London.

[17] Long van, N., A. Soubeyran, 2000, "Existence and Uniqueness of Cournot Equilibrium: A Constraction Mapping Approach", Economics Letters, 345-348.

[18] Maskin, E., J. Riley, 1984, "Optimal Auctions with Risk Averse Buyers", Econometrica, 52, 1473-1518.

[19] Milgrom,P.,R.J. Webber: 1982, "The Value of Information in a Sealed Bid Auction", Journal of Mathematical Economics, 10, 105-114.

[20] Levitan, B. Shubik, 1979, "Moral Hazard and Observability", Bell-Journal-of-Economics, 10(1), 74-91.

[21] Milgrom,P., R.J. Weber "Distributional Strategies for Games with Incomplete Information", Mathematics of Operations Research, 10, 619-632.

[22] Osborne,M.J.,Carolyn Pitchik, 1986, "Price Competition in a Capacity-Constrained Duopoly", Journal of Economic Theory, 38, 238-261.

[23] Shapiro, C. 1986, "Exchange of cost information in oligopoly" , Review of Economic Studies, 53, 433-446.

[24] Spulber, D.F. 1995, "Bertrand Competition when Rivals' cost are Unknown" , The Journal of Industrial Economics, 43(1), 1-11.

[25] Vives, X., 1984, "Duopoly Information Equilibrium: Cournot and Bertrand" , Journal of Economic Theory, 34, 71-94.

[26] Vives, X., 2000, "Oligopoly Pricing: Old Ideas and new Tools" , MIT Press, London.