On Chicken, War and, Other Auctions *

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Abstract

This paper analyzes varieties and similarities of the simple Chicken Game and the War of Attrition Game. We extend the Chicken Game to a continuous environment and find a simple parameterization that connects the two models. The War of Attrition Game typically has many equilibria. By contrast, the continuous Chicken Game does not have any Nash Equilibrium - neither in pure nor in mixed strategies. In an extension we mix the two models. We emphasize that for continuous games the definition of a Nash Equilibrium is not weak enough to guarantee equilibrium existence.

Keywords: Chicken Game, War of Attrition, Nonexistence of Equilibrium, Fully Mixed Strategies

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1 Introduction

In this article we try to investigate two introductory economic models: Basic Game Theory deals with the simple Chicken Game, introductory Industrial Organization discusses the War of Attrition Game.

The Chicken Game is very simple and the story behind is well known beyond economics: Two young guys, we call them Brad and Pitt, compete for the favors of young Jennifer. To demonstrate their love they agree in playing the following morbid game. Each has to take his car and to drive towards the rival. The first person to swerve is declared the Chicken. The other one is the hero, capturing Jennifer's affection. If none swerves, they crash and both guys die in the accident.

There are several reasons for the presentation of this game to young economists: The originality and the convincing story, the payoff-selection problem and the simple computation of the mixed strategy equilibrium. The originality is well known. In a famous taxonomy Anatol Rapaport and Melvin Guyer (1966) found 78 different 2x2 games. The chicken game is one of them. The story can be emphasized with different examples. A widely accepted application for the Chicken Game in real life is the arms race with nuclear weapons. Two countries can either invest in nuclear weapons or not. If one single country has a nuclear weapon, it has absolute power and its payoff is (or should be) higher than the opponent's. If none invests in nuclear power, none of them can be declared the chicken, power is equal and none has to fear an attack. If both invest in a nuclear technology, resources are wasted and the two countries are equal. Both spent a lot of money, have to fear a nuclear attack, but none has any military advantage. This outcome is the worst case for both players.

Military sciences examined this game theoretic model for decades. But also other sciences used the Chicken Game to analyze real world phenomena. Social dilemmas like the natural resource problem, especially the consumption of petrol, have been modelled by a (repeated) Chicken Game.

A second reason to present the Chicken Game in introductory courses may be the payoff problem. What is the payoff if someone dies? Is it allowed to set a payoff equal to minus infinity? What are the effects if a payoff is unbounded? Why should an equilibrium concept sustain a crash in equilibrium? These questions are not without controversy. A lot of students do not accept if an economists says payoff in the accident case is zero, not minus infinity. Others are persuaded that a solid equilibrium concept must not allow for a crash in equilibrium. The popularity of the Chicken game may be due to this controversial payoff-selection problem.

A third reason is the fact that this game has a simple mixed strategy equilibrium. Moreover, even though the game has equilibria in pure strategies, the mixed one may be the only one that is reasonable. This mixed strategy equilibrium can be easily computed, but students wonder that a crash occurs with positive probability whenever payoffs in the crash case are finite.

Of course, gains from the analysis of the simple Chicken Game are exhausted. The reason why we want to discuss the Chicken Game once more is that the story of the game can be modelled in a much more realistic way. The Chicken Game is in fact a dynamic one. Therefore, the story and the game theoretic modelling do not really coincide. The two guys pull out and time elapses. Hence, the two players have to decide when to exit (or whether they want to exit or not at every moment, respectively). The consequences of these dynamics are not innocuous. They have serious impacts on equilibrium behavior.

The Continuous Chicken Game is more general than the ordinary one. Our model applies to every problem that can be modelled with the simple Chicken Game. Later, we will discuss examples that can not be modelled by the simple 2x2 Game. An important category is especially exit games.

The idea of exit in a dynamic game leads us to the second model this paper is about. Introductory courses in Industrial Organization discuss the well known War of Attrition game. To explain the model, we deal with a very simple two firm version. Consider a duopoly market with homogenous goods and fixed costs. Firms compete in prices and have to pay an interest rate on fixed costs. The Bertrand result implies that if both firm decide to be active at a certain period, they both set prices equal to marginal costs. So firms make losses at any point in time. These losses increase exponentially because of compound interest. But firms can leave the market.[1] If a firm leaves the market, payoff is zero from there on, which is better than suffering losses. The reason why firms may stay in the market is the possibility that the opponent exits first. In this case, the remaining firm stays as a monopolist, sets supracompetitive prices and makes positive profits if fixed costs are not too high. In this war of attrition firms now have to decide when it is optimal to leave the market.

There are various stories for the model. War of attrition was originally developed in biology. Two animals of the same species compete for food. The competition is costly since both have to spend energy. As soon as one animal stops fighting, the other one gains the meal. Biologists analyzed how long the animals fight for the food. Examples in Biology are Maynard Smith (1974) or Riley (1980).

Military scientists used the War of Attrition Game to model nuclear arms races.[2] As long as several countries have a nuclear weapon, all of them bear costs of mutual deterrence, security, research and development and the risk of a military attack. In principle each country would prefer not to have nuclear arms - but if and only if the other countries do not have any either. If only one country disarms unilaterally, the position of this country is worse. The issue is how long countries bear the costs and whether unilateralism is possible.

However, a lot of economists analyzed this setup for economic purposes. Different real life examples are discussed in the literature and made the model very

[1]For simplicity we assume that firms cannot reenter the market or that reentering costs are prohibitively high respectively.

[2]As nuclear arms races have been modelled by War of Attrition and the Chicken Game, we can deduce that there must be a relationship between the two models.

popular. The idea of exit in a duopoly was first analyzed by Fudenberg and Tirole (1986). Alezina and Drazen (1991) modelled the conflict over government budgets with a War of Attrition Game.

The model is also important for auction theory. The setup coincides with an all pay auction. Typically, we find a chapter about war of attrition in every auction textbook, e.g., Milgrom (2004). The reference for our paper is the article by Hendricks, Weiss and Wilson (1988). They give a concise solution for the full information case. Important additional papers for War of attrition are Bliss-Nalebuff (1984) or Hendricks-Wilson (1985).

For our simple duopoly exit game the most important real world example is the story of Kodak and Polaroid in instant photography. Others were Microsoft vs. Netscape for Web-browsers or newspapers in several cities. In all these examples some firms toed the line to suffer losses, hoping that the opponent leaves earlier. The idea of so called "cut-throat competition" or predatory pricing is widespread especially in the German speaking area. The phenomenon is relevant for firms but especially for competition authorities and politicians. Since firms waste resources during their war of attrition the outcome is usually inefficient. Some economists argue, therefore, that firms should not be allowed to set prices below average costs. We have to reply that this argument is not valid if we allow for reentering. If reentering is possible a firm can set prices equal to marginal costs or higher, making positive profits if the other opponent also sets high prices. If the opponent sets prices below average costs the firm faces zero demand and zero profits. But there is no reason why the firm should drop out of the market. It will reenter as soon as the other firm will set price at or above marginal costs. Hence, in the long run there is no reason for cut-throat competition if reentering is possible.

The model can easily be extended to more players. The theoretical solution is due to Bulow and Klemperer (1999). But they state that very often markets with N firms reduced quite fast to "a two-horse race". This is one of the reasons why we will analyze just the two player case. Other reasons are apparently the simplicity and the fact that we have not seen a multi-person Chicken Game yet.

This paper is about two major issues. The first one is the basic connection between the Chicken Game, the War of Attrition game and auction theory. The second is the nonexistence of any Nash Equilibrium in the Continuous Chicken Game.

The Chicken Game and the War of Attrition game have a lot in common. Both games require exactly one player to exit to end. The exit time is the only strategic variable. In this paper we find a generalized model - a real meta-model. The crucial point is how the profit of each firm depends on time from the start to the point where one firm exits. Assume two firms start at zero. In a War of Attrition game firms' losses increase the longer the firms are in the market. This leads to the phenomenon that firms exit often at the very beginning.

The assumption that the payoff is smaller the longer a players stays in is not appropriate in the case of a Continuous Chicken Game. Here it makes a difference whether the loser exits immediately or right before the two players crash. Despite he is the Chicken in any case, it is obvious that social standing of the early leaver will be much lower than for a guy who left only 1 second before the cars crashed. Therefore, we assume that the payoff is increasing over time.

In a first stage we discuss the model if a player's payoff is monotone over time and has a constant sign. In a second stage we mix the Chicken and the War of Attrition game and assume that the time-dependent part of the payoff function evolves according to a Standard Wiener Process.

The paper is organized as follows. Section two reexamines the simple Chicken Game and explains the model. Section three examines the War of Attrition game with its two approaches "commitment case" and "continuous decision" case. Section four does the same for the Continuous Chicken Game. Section three and four also link the two models to auction theory. Section five discusses a mixture of the two models - the Wiener Process case. Section six concludes.

2 The General Model

In the ordinary Chicken Game each player has two strategies. Swerve or do not swerve. Figure 1 shows possible payoffs.

	Swerve	Drive on
Swerve	5 / 5	0 / 100
Drive on	100 / 0	-1000/-1000

Figure 1: Payoff Matrix of the Simple Chicken Game

To keep the game simple we assume it to be symmetric. This game obviously has two equilibria in pure strategies (Swerve, Drive on), (Drive on, Swerve) and one in mixed strategies. In the mixed strategy equilibrium each player gives way with probability $\frac{1000}{1095}$. Expected payoff in the mixed strategy equilibrium is equal to 4.57. A crash occurs with probability 0.007.

There are several reasons that this model is not satisfactory. Most apparently, the discrete modelling of a continuous decision process seems unappropriate. Therefore, we set up the model in continuous time. The second modification is based on the fact that there is a difference between a sidestep at the beginning and a sidestep shortly before a crash would occur. Only one guy can be Jennifer's boyfriend. But it seems natural that a loser who drove a long time is not as big a Chicken as the loser who exited at the very beginning. The setup in the continuous chicken is straightforward. See first figure 1.

Player 1 starts at point 0, player 2 starts at point 1. They drive towards each other. As soon as one player swerves (exits) the game is over. The one who swerved is declared the chicken. The other one is the winner. If none of the two player exits before time T (e.g. in the middle), they crash. This is the simple

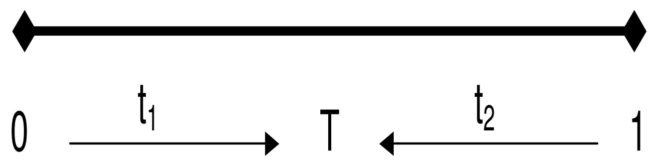

Figure 2: Chicken Game

story - exactly the same as in the ordinary chicken.

But we are not only interested in the Chicken story. We want to analyze a more general model. To set this model up we explain the general setting of the game a little bit more abstract:

Two players start a game at the same time $t = 0$. The only move the players have to make at time t is to simultaneously announce when they want the game to end. This corresponds to the decision when they want to exit (commitment case) or whether they want to exit or not, given the opponent did not exit yet (continuous decision making).

Both players know that the game cannot take more than time T, i.e. their announced exit time must be higher than T. This time T is common knowledge. If none of the players decides the game to end before T, both will get a payoff of $-C < 0$. This is the case when Brad and Pitt die in the accident. Putting the things together: they have to choose a time $t \in [0, T]$ when they want the game to end. Depending on the elapsed time both players receive a payoff $\int_0^t \delta(w)v(w)dw$, where $\delta(t)$ is a discount factor and $v(t)$ is a time dependent payoff component. At the moment we do not make any further assumptions on $v(t)$. Later, we will show that this function is the key that connects the Chicken Game to the War of Attrition game. The last remaining player in the game receives an additional special payoff of discounted $G > 0$. The first player who wants the game to end does not get an extra payoff. With other words: Players do not want to exit because the last remaining player receives the “prize” G. This G is the valuation of Jennifer’s affection.

Despite this “prize”, both players get the payoff $v(t)$. For this reason the payoff depends on t. Money or utility flows have to be discounted by the interest rate r. If one player exits at a time $\tilde{t}$ the game is over. Both get the payoff $\int_0^{\tilde{t}} \delta(w)v(w)dw$ and the winner gets additionally the prize $\delta(t)G$. The payoff function for firm i at time 0 is therefore:

$$U_{i0}(t_i, t_j) = \begin{cases} \int_0^{t_j} e^{-rw} v_i(w)dw + e^{-rt_j} \int_0^{\infty} e^{-rw} G dw & \text{if} \quad t_j < t_i < T \\ \int_0^{t_i} e^{-rw} v_i(w)dw & \text{if} \quad t_i \leq t_j < T \\ -e^{-rT} C & \text{if} \quad t_i = t_j = T \end{cases}$$

The games incentives crucially depend on the payoff part $v(t)$. We will show later that $v(t) \geq 0 \ \forall \ t$ implies the continuous Chicken Game and $v(t) \leq 0 \ \forall \ t$ the War of Attrition Game. Since the game is over after one player exits, the game lasts exactly the minimum of t_i and t_j. It is important to note that the GAME (the strategic interaction) ends at this point not the story. The remaining player will have a never ending payoff $\frac{G}{r}$ at this time.

For our parameterization the payoff of the winning firm is independent of the own strategic variable but increasing in the opponent’s. This makes sense because for the winning player it would be better if the winning decision was narrow. The winner is better off if he can say that he won against a tough guy.

To simplify notation we make the following definitions:

$$H_i(t_i, t_j) := \int_0^{t_j} e^{-rw} v_i(w)dw + e^{-rt_j} \frac{G}{r}$$

$$L_i(t_i, t_j) := \int_0^{t_i} e^{-rw} v_i(w)dw$$

$H_i(t_i, t_j)$ is equal to the net present value of the expected payoff at time zero if firm i announces the higher t, i.e. if firm i wins. Analogously $L_i(t_i, t_j)$ is the payoff of firm i if it exits first, i.e. if player j is the winner.

There will be mixed strategy equilibria. Assume firms play (fully) mixed strategies $F(t)$. It is well known from the literature that these distributions do not have mass points, except at the infimum and the supremum of their equilibrium distribution $t = \hat{T}$. We will also see in lemma 1 that for our parameterization masspoints occur only at the supremum of the equilibrium distribution. We denote the probability at the supremum as $\Delta F(t) = \alpha$ (if it exists). This implies that the probability that both firms leave at the same time $t_i = t_j < T$ is 0. With these considerations we can now write the expected payoff of firm i as:

$$E_{i0}(U_i) = \int_{t_i}^{\hat{T}} L_i(t_i)dF(t_j) + \int_0^{t_i} H(t_j)dF(t_j) - \alpha e^{-r\hat{T}}C$$

where $\hat{T}$ is the point where each firms leaves the market with probability 1, i.e. the supremum of the equilibrium distribution. α may be zero.

We try now to extract the two special cases from this model. We will show that depending on $v(t)$ equilibrium behavior is completely different.

3 The War of Attrition Game

To receive a War of Attrition Game we make the following assumptions.

Assumption 1 : $v(t) \leq 0 \; \forall \; t$, $C, G \in \mathcal{R}^+$ with $C > G$.

To classify the nature of the War of Attrition game we do not make any further assumptions on $v(t)$. Especially we do neither assume that $v(t)$ must be continuous, nor do we have to make an assumption about any derivative of $v(t)$.

As our aim is not to discuss technical things we keep things as simple as possible. We assume

Assumption 2 : $v(0) = 0$, $v(t)$ continuously differentiable on $[t, T)$.

Compared to the standard model time T when the war ends and the punishment factor $-C$ are new. But both entities show up in the literature. Hendricks et al. discuss the full information case with a duration of 1. But this is only a standardization as we will see later. They have also a discontinuity at time T. But they do not discuss our case in detail.

We assume, although not necessary for our analysis, that $C > G$ in order to stress the punishment. Our story is almost the same as in the standard model. Economists and biologists accept that a war of attrition has a given duration. It seems plausible that if none of the two players finished before the end of game both get a punishment and the payoff is negative. In Hendricks et al. there is an example where firms that are active at T receive a zero payoff instead of a positive payment.

The punishment factor at T may exist for several reasons. It may be that two entrepreneurs cannot make infinite losses. Wifes and husbands can not always be adamant to go to the opera or to boxing matches. Farmers cannot always seed the same thing and so on. T stands for the "last straw that breaks the camel's back". As both firms being active at T corresponds to a loss of any chance to win the monopoly rent this metaphor is indicated. An additional example could be the collapse of a common. Since the farmers made too much use of it at a time T the common collapses and both shepherds go bankrupt.

Finally, we could add the theory of bargaining. Consider a situation in which two agents have to split a dollar. They have a certain time to find an agreement. If they do not find one, both get a zero payoff. A time T when all players loose their positive profits if they did not agree is well known in bargaining literature or in principal agent theory. Of course, this setup is not in the sense of the Rubinstein - Bargaining model. But we find models where players are punished if they do not find an agreement in a given time especially in political economics.

Even more relevant are such setups for reality. Since time for a debate is limited a bill has to be accepted within the stated time or the motion is defeated. A defeated motion may be very costly for all the parties.

The time T is relevant if and only if none of the firms exited. So the existence of T is symmetry independent.[3] If the valuations $v(t)$ are not equal we are in the asymmetric type model. The player with the bad type typically exits immediately. As mentioned above it is not our aim to give a full theory of War of attrition. We want to make the point that Chicken and War of attrition are special cases of the same game and want to give the basic insights.

We now have to split the model into two different approaches. First, we could assume that each player has to choose his exit time at $t = 0$. Each can perfectly commit to exit at this time t. Second, we could assume that each player chooses his exit time at any t given that the opponent did not exit yet. We call the model where each player has to choose its exit time at the very beginning the "Commitment Case". The model where each player decides at every t whether he wants to exit or not, given the opponent did not exit yet, the "continuous decision" case. We first want to analyze the commitment case:

3.1 Commitment Case

The commitment case stresses the noncooperative character of the game. In the commitment case each player has to choose his exit time at the very beginning. He can not alter his decision. We can imagine that each player has to submit his exit time in an envelope. We will discuss the strategic equality to a sealed-bid second price all-pay auction at the end. We assume that the two players cannot coordinate their decisions and collusion is prohibited. The players open the envelopes and declare the winner.

The set of feasible actions is $t \in \mathcal{T}$ with $\mathcal{T} = [0, T]$. We are interested in a Nash equilibrium.[4] A strategy is a time t when a player decides to exit. The players cannot make an exit decision conditional on the opponent's exit. Each

[3] It is crucial that the system collapses if and only if both players are active at T.

[4] Subgame perfection is not restrictive for our examples. See Hendricks et al..

player accumulates a negative payoff during the elapsing time t. He does so in order to win the "prize" G.

But we have to point out that our analysis includes the case where a player has to compute his equilibrium strategy at any t. It turns out that the formal analysis would be exactly the same. Assume a player chooses hits time to exit at every t. In this case anything that happened before does not play a role - $\int_0^t \delta(w)v(w)dw$ is sunk. Hence, the player faces a new problem with well defined $v(t)$, G and $-C$. The only thing that has changed is the smaller T.[5] At any t the players compute their equilibrium strategy. As a part of their fully mixed strategy equilibrium they compute a probability to exit. When time t changes, the equilibrium distribution will change too and the player will play a new equilibrium strategy.

We now state the result for the once-and-forever decision War of Attrition Game with punishment. A firm is at $t = 0$ and has to decide how long it wants to stay in the market. This game obviously has two simple pure strategy Nash Equilibria: $\{0, T\}$,$\{T, 0\}$. Whenever the opponent will never exit (will always stay up to T) the firm should exit immediately. This is true even for small T. If the opponent exits immediately this firm will win whenever its t is positive. The firm's payoff will be independent of its strategy. But as it wins the rent G immediately any $t > 0$ gives the same payoff without any costs. Hence, the firm can play T and we have established a PSNE. But we will see in the theorem that there are other PSNE. To derive the mixed strategy equilibrium is a little bit harder. The mixed strategy equilibrium will be a fully mixed strategy $F(t)$ on the set $(0, T)$. To compute the mixed strategy we need the following three lemmata first.

Lemma 1 : A non-degenerated symmetric mixed strategy equilibrium $F^*(t)$ can not have an atom at $t = 0$.

[5] We cannot conclude that we have serious time inconsistency problems. As new information occurs (the opponent did not exit) it is clear that each player should behave differently.

Proof of Lemma 1 : The expected profit of a firm must be equal for any argument where the distribution $F(t)$ has a positive value. A mass point at $t = 0$ implies that the expected payoff must be zero at any t where $F(t) > 0$. Suppose firm j plays 0 with probability α. In a symmetric equilibrium both firms must have an expected payoff equal to zero. But if firm j has an atom at $t = 0$ firm i can get a positive expected payoff by playing ε with probability 1. The expected profit will be $\approx \alpha\,(v(0) + G) + (1 - \alpha)v(\varepsilon) > 0$. ■

Lemma 2 : In a nondegenerated symmetric mixed strategy equilibrium the expected payoff of each bidder is zero.

Proof of Lemma 2 : Elementarily the equilibrium expected payoff can not be negative. Suppose the expected profit would be positive. This implies $F'(0)$=0. But at $\underline{t} > 0$ each firm wins with probability 0 which implies that the expected profit at $\underline{t} < T$ is not positive. This implies that we cannot be in a mixed strategy equilibrium.■

Lemma 3 : The infimum of the support of the equilibrium distribution $F(t)$ must be $\underline{t} = 0$. The supremum must be $\bar{t} = T$.

Proof of Lemma 3 : Suppose $F(t)$ would be an equilibrium distribution with support $(\underline{t}, \bar{t})$ and $\underline{t} > 0$. In this case each player has an incentive to deviate by setting positive mass on 0. Instead of playing strategies around $\underline{t}$ and loosing with a high probability each player should play 0, where he looses with probability one, but does not have to bear the costs $v(\underline{t})$. Suppose there would be an equilibrium distribution with $\bar{t} < T$ In this case each player could do better by putting positive mass on $\bar{t} + \varepsilon$ In this area he wins with probability 1 and the expected profit will increase. ■

We are now able to state the theorem:

Theorem 1 : Define $\hat{T}$ implicitly by $H(\hat{T}) = 0$. The War of Attrition Game with once-and-for-ever decision making has the following Nash Equilibria: If $T > \hat{T}$ there is a continuum of Nash equilibria. Any combination of strategies with $(0, t)$ or $(t, 0)$ constitutes a NE if $x \geq T$.[6] Additionally there is one equilibrium is in mixed strategies. The mixed strategy equilibrium $F^*(t)$ is atomless on the set $(0, \infty)$ and must fulfil the differential equation:

$$\frac{L'(t)}{L(t) - H(t)} = \frac{F'(t)}{1 - F(t))}$$

and the initial condition

$$F(0) = 0$$

If $T \leq \hat{T}$ the game has the two equilibria in pure strategies $(T, 0)$,$(0, T)$. There is also a mixed strategy equilibrium with .

$$\frac{L'(z)}{L(z) - H(z)} = \frac{F'(z)}{1 - F(z))}$$

with $t = -\frac{z}{z-4}$.

Additionally, there are mixed strategy equilibria where both firms have a mass point at T. This equilibrium is characterized by the same differential equation and the same transformation but different initial conditions:

$$F(0) = 0$$

$$-\alpha^2 C + \alpha(1 - \alpha)(G - v(T)) = 0$$

$$\lim_{\varepsilon \to 0} F(T - \varepsilon) = 1 - \alpha$$

Proof of Theorem 1 : If firm $i's$ opponent stays up to $\hat{T}$ the profit of firm i can not be positive. Since expected profit of an immediate exit is 0, a firm

[6] We also have a continuum of mixed strategy equilibria where one player exits immediately with probability one and the other mixes on the set $\mathcal{A} = [\hat{T}...\infty)$. But t's that are higher than $\hat{T}$ are dominated in the sense that if the other firm makes a mistake with probability ε, it is better to play $\hat{T}$.

would never stay up to the point where the expected profit is negative. Hence, all strategies $t > \hat{T}$ are weakly dominated. If T is sufficiently large i.e. T goes to infinity we are in the standard War of Attrition Game. T does not play any role. PSNE are straightforward. If firm j stays up to $\hat{T}$ it is optimal for i to exit immediately. If the opponent exits immediately firm i will win with any $t_i \neq 0$. Since min (t_i, t_j) will be 0 in any case player i is indifferent between any $t_i \geq \hat{T}$. Hence, he can play $t_i = \hat{T}$ in equilibrium.

The mixed strategy equilibrium is slightly more complicated. If a firm plays a fully mixed strategy $F(t)$ in equilibrium, then it must have the same payoff at every argument t. Expected payoff is equal to

$$\int_{t_i}^{T} L(t_i)dF(t_j) + \int_{0}^{t_i} H(t_j)dF(t_j) - \alpha e^{-r\hat{T}}C = 0$$

Differentiating with respect to t_i, equating to zero and rearranging the terms yields the result. Our solution coincides with the solution by Hendricks et al..

As the equilibrium distribution cannot have a mass point at $t = 0$ by lemma 1 we have the initial condition $F(0) = 0$.

If $T \leq \hat{T}$ pure strategy equilibria are straightforward. We know that on the set $[0...T)$ where the equilibrium distribution is atomless the players choose t according to the same differential equation as in the case $T > \hat{T}$. But the initial conditions differ. We further know that the expected payoff must be 0. We know that the supremum of the equilibrium distribution must be T. There are two possibilities to get 0 in equilibrium: Either there is a mass point at T or there is not. If there is no mass point at T things are easy, we can apply the upper differential equation with the initial condition $F(0) = 0$ and the stated transformation[7].

If there is a mass point at T things are slightly different. Suppose in equilibrium each firm plays T with probability α. The expected profit of firm i if it plays T is $-\alpha^2 C + \alpha(1-\alpha)(G - v(T))$. This must, of course, be zero. So we have

[7]For the transformation see Hendricks et al..

the probability α. As the initial condition $F(0) = 0$ still applies we have two conditions that must be fulfilled. Please note that this kind of equilibrium exists if and only if both conditions hold. Otherwise there would be no equilibrium with a mass point at T. ■

This simple War of attrition model is strategically equal to a sealed bid second price all pay auction. Let us first assume that T is very large. In this case there is a one-to-one correspondence to the basic story. The valuation of the prize is equal to the present value of the monopoly rent. The bids are equal to the exit times t_i, t_j. These times have to be transformed by the function $v(t)$ that is common knowledge. So it does not matter whether players choose t or $\int_0^t \delta(w)v(w)dw$. Both firms have to pay the transformed bids of the firm that bids less . The one with the higher bid wins the auction and the prize $\frac{G}{r}$.

The auction analogy is more complicated to install if T is not large. In this case the auction would be modified with a punishment factor if bids are too high. It is hard to find a convincing story to sustain such a punishment factor but we try to do it: Assume an old grandmother has a worthy goblet that she wants to give to one of her two sons. On the one hand she thinks that the goblet should go to the son that values it highest. On the other hand she would not like this goblet to be a cause of conflict. Finally she does not want that the sons pay too much for the goblet. For this reason she designs the following auction. Both sons have to participate in a sealed bid second price auction. Both must not bid more than a certain value - the maximum value the grandmother sets. If both of them set the maximum value the grandmother does not know to whom she should bequeath it and decides to give it to a charitable trust. Since the two sons do not want to give the goblet to foreign hands they suffer nonmonetary losses in this case. We have the value of the goblet equal to the net present value of G. We have the all pay auction with bids equal to the integral over $v(t)$ and we have the punishment factor $-C$.

Typically an auction implies that the higher a bid the higher the probability to win. This is not the case if we include T. With T we have a compact strategy space. As bids cannot be too high we have something that we could call a target auction.

We now give the result for the following example. $v(t) = -t^2, G = 100, r = 0, C = 100$. If T goes to infinity (i.e. does not exist) the mixed strategy equilibrium distribution is

$$F(t) = 1 - e^{\left(-\frac{t^2}{100}\right)}$$

We see that in equilibrium each firm plays $t's$ that are higher than $\hat{T}$ with positive probability. The reason for that is that the payoff of the winning firm is independent of the own strategy. Only high $t's$ set the opponent indifferent between the strategies.

If T is equal to 4 the equilibrium distribution is

$$F(t) = 1 - e^{-\left(\frac{t^2}{100(t-4)^2}\right)}$$

We finally give an example of an equilibrium with a mass point at T. Suppose $v(t) = -t^2, G = 100, r = 0, C = -100, T = 0.9903$. The mixed strategy equilibrium distribution is

$$F(t) = \begin{cases} 1 - e^{\left(-\frac{t^2}{100}\right)} & \text{if} \quad 0 \leq t < T \\ 1 & \text{if} \quad t = T \end{cases}$$

Each firm plays T with probability 0.9899.

3.2 Continuous Decision Making

The continuous decision case stresses the noncooperative character of the game. Each player has to choose his exit probability at every time t given the opponent did not exit yet. We will discuss the strategic equality to an ascending open all-pay auction at the end.

The set of feasible actions at date $t_0 \in \mathcal{T} = [0...T)$ is $p \in [0,1]$. We are interested in a subgame perfect Nash equilibrium. A strategy is a sequence of exit probabilities that each player announces, given the opponent did not exit yet. The players make exit decisions conditional on the opponent's exit.

The equilibria can be characterized as follows:

Theorem 2 : The War of Attrition Game with continuous decision making has the following symmetric subgame perfect Nash equilibria for a game starting at t_0. Define $\hat{T} \equiv \int_{t_0}^{\hat{T}} \frac{\int_0^t -e^{-rw}v(w)dwr}{e^{-rt}G} = 1$. If $T > \hat{T}$, the symmetric war of attrition game has three Nash equilibria. Two are in pure strategies: at each t_0 choose (exit, stay), (stay, exit). Additionally there is one equilibrium in mixed strategies. $p(t) = \frac{\int_0^t -e^{-rw}v(w)dwr}{e^{-rt}G}$ for $t \in [t_0...\hat{T})$.

If T is small there are three Nash equilibria. Two are in pure strategies (exit, stay), (stay, exit), one is mixed strategies $p(t) = \frac{\int_0^t -e^{-rw}v(w)dwr}{e^{-rt}G}$ for $t \in [t_0...T)$.

Proof of Theorem 2 : If T is high we are in the standard War of Attrition Game. The pure strategy Nash equilibria are straightforward. If the opponent stays in it is optimal to exit immediately. If the opponent exits with probability 1 the player will win if he stays in whenever $t_0 < \hat{T}$. The mixed strategy equilibrium is slightly more complicated. If a firm plays a mixed strategy $F(t)$ in equilibrium the expected payoff of staying in the market must be equal to the expected payoff if the firm exits. Assume the opponent leaves with probability p. Hence, $\int_0^t e^{-rw}v(w)dw + pe^{-rt}\frac{G}{r} = 0$ since exit implies no additional payoff and v(t) is sunk at this time. Solving for p we obtain the result. The domain $(0, \hat{T})$ is straightforward.

If T is small i.e. $T \leq \hat{T}$, pure strategy equilibria are straightforward. It is the same argument as above. Let us compute the mixed strategy equilibrium. Assume a firm is at a point t. In a non-degenerated mixed strategy equilibrium each player has to set the opponent indifferent between exit and stay. Both players have to set the other indifferent at one special t and just for the next moment. As every t on the set $[0...T)$ has a neighbor the punishment factor $-C$ is irrelevant for any t. As at T the active firms do not have to choose whether they want to exit or not $t = T$ is not in the strategy space. Hence, the equilibrium in the T small case is exactly equal to the case where T is large. Because of the open set firms are indifferent at every t and they never have to fear that they reach T. ■

It is important that subgame perfection typically does not restrict the equilibria. As Hendricks et al. argue this is the case only if one player exits immediately. This is the not the case for the mixed strategy equilibrium.

Let us investigate this result from an other point of view. Suppose this game would be discrete. Since the game does not have an end, we can not use backward induction. But at every starting point of a new subgame, we face the same game as at the very beginning. The optimal behavior at this starting point is history independent and the game is stationary. The probability that both players enter the next subgame is always positive. The entire reason that they enter the next subgame is that the other exits with a positive probability. As at every point infinitely many subgames follow, the probability to reach the point where a crash occurs is 0 for every t.

This simple War of attrition model with continuous decision making is formally equal to an ascending all pay auction. We can take the same example as for the once-and-forever decision case. The difference is obvious. As in each period the players learn whether the opponent exited or not. This information alters equilibrium behavior. As we are in a continuous decision environment the difference between a second-price and a first-price auction does not matter.

4 The Continuous Chicken Game

To get obtain a continuous Chicken Game we make the following assumptions.

Assumption 3 : $v(t) \geq 0 \; \forall \; t$, $C, G \in \mathcal{R}^+$, $C > G$

Please note that we do not make any further assumptions on $v(t)$.

We can expand the formal analysis of the continuous Chicken Game to other stories. We have to change the War of Attrition story only a little bit. Consider a duopolistic market with increasing demand. For example we could take the market for telecommunication. The two firms compete in quantities. Therefore, the two firms make positive profits. Because of increasing demand profits are increasing over time. Both firms know that there will come a time T when a new technology will be implemented. From this time on, two active firms will make losses in the market. If only one firm is active it will set the monopoly price and will earn positive profits. The monopoly rent in T will be equal to G, the loss in the case where both firms are active at T is $-C$ and the increasing cournot profit is $v(t)$. Time T may also be the time where demand is so high that a new (e.g. a foreign) player enters the market. As entrants very often are more efficient it seems plausible that the incumbents bear losses after the new player entered.

We will see that the equilibrium analysis is unsatisfactory. As in the War of Attrition case it is possible to separate two different cases.

4.1 Commitment Case

We can analyze the model when both players decide at the very beginning when they want to exit. This "commitment case" is more in the sense of the static 2x2 Game. It stresses the noncooperative aspect of the game. It is again in some sense a "sealed-bid" auction. We will come to this later.

The set of feasible actions is $t \in \mathcal{T}$ with $\mathcal{T} = [0, T]$. We are interested in a Nash equilibrium. A strategy is a time t when a player decides to exit. The players cannot make an exit decision conditional on the opponent's exit. Each player accumulates positive payoff during the elapsing time t. This creates incentive to exit as late as possible.

Theorem 3 : The continuous Chicken Game with commitment does not have any Nash Equilibrium - neither in pure strategies nor in mixed strategies.

Proof of Theorem 3 : It is easy to prove that this game does not have a pure strategy Nash Equilibrium. Assume a pure strategy combination (t_1, t_2) would be an equilibrium. This implies that one player has to play $t_i = T$ since otherwise the one with lower t should increase its t in order to overbid the competitor and to win G. If one player plays T with probability 1, best answer would be to play the highest possible t that is lower than T. But the problem max t, s.t. $t < T$ does not have a solution.
The nonexistence of a mixed strategy Nash Equilibrium is slightly more complicated. Assume a strategy combination $(F_1^*(t_1), F_2^*(t_2))$ is an equilibrium. If no player puts positive mass on T, this strategy can not be an equilibrium since the other player would choose T with probability 1. Assume one player puts positive mass on T. Then it is optimal for the other player to play $T - \varepsilon$ with probability one and ε as small as possible. With other words: Assume firm 1 plays the mixed strategy T with probability α and $T - \varepsilon$ with probability $1 - \alpha$. Then firm 2 will never put positive mass on T for any α since the payoff when it plays for example $T - \frac{\varepsilon}{2}$ would be higher for any α. Firm 2's maximization problem does not have a solution either. This completes the proof. None of the two player can set the other indifferent on any set $\tau \in [0...T]$. ■

This is a completely unsatisfactory result. However, the underlying story is not absolutely new. The game by Sion and Wolfe (1957) has a similar strategic reasoning. But the example is interesting for practical purposes. The story shows that nonexistence can be established very easy in reality and that the

Nash-Equilibrium is not weak enough in order to assure equilibrium existence.

We have to point out that in the continuous Chicken game the players can not force the opponent to mix his strategies. Of course anyone could argue that this is a dirty trick: the PSNE equilibrium is $(T, T - \varepsilon)$ with ε as small as possible. But this is not true! A strategy is a complete contingent plan. In our case this must be a time t. But you can not say to a player what t he should choose. A Nash equilibrium does not allow to play a supremum - it must be a maximum.
We try to explain this in other words. In a discrete game where players have to decide whether they want to exit at period $t = 1, 2, 3..., T - 1, T$, there would be three equilibria. Two in pure strategies $(T, T-1)$ and $(T-1, T)$ and one in mixed strategies where both players mix between T and $T - 1$. This property does not carry over to the continuous case. Sure, every player would like to play $T, T - 1$ and a mix of these two pure strategies. But this is not possible in a continuous environment since (T-1) does not exist or can not be localized respectively.

Like the War of Attrition Game the continuous Chicken Game can be analyzed as an auction. While the War of Attrition Game with commitment is is a sealed-bid all-pay auction, we have here a sealed-bid all-pay auction with negative prices. We could call it a sealed-bid all-win auction with punishment. The auction analogy is more for technical purposes than for an economic intuition. We do not know any real-world situation where a seller should choose such a sealed bid all-win auction with punishment. We want to remark that this auction may be characterized as a target-auction as well. Each player tries to bid as close as possible to a given target.

4.2 Continuous Decision Making

If players have to decide at every t whether they want to exit or not, the problem is different. The continuous decision game stresses the dynamic aspect of the Continuous Chicken Game. Players have to state at any t whether they want to exit. The set of feasible actions at date $t_0 \in \mathcal{T} = [0, T)$ is $p \in [0...1]$.We are interested in a Nash equilibrium. A strategy is a sequence of exit probabilities that each player announces, given the opponent did not exit yet. The players make exit decisions conditional on the opponent's exit.

Theorem 4 : The Continuous Chicken Game with continuous decision making does not have an Nash equilibrium, neither in pure nor in mixed strategies.

Proof of Theorem 4 : Assume a firm is at a point t. In a non-degenerated mixed strategy equilibrium each player has to set the opponent indifferent between exit and stay. Both players have to set the other indifferent at one special t and just for the next moment. If each player plays a mixed strategy $F(t)$ in equilibrium the expected payoff of staying in the game must be equal to the expected payoff if the player exits. Assume the opponent leaves with probability p. Hence, $\int_0^t e^{-rw} v(w) dw + pe^{-rt} \frac{G}{r} = 0$ since exit implies no additional payoff and v(t) is sunk at this time. Solving for p we obtain that p should be negative. Hence the exit probability has to be zero. As every t on the set $[0...T)$ has a neighbor, the punishment factor $-C$ is irrelevant for any t. As at T the active firms do not have to choose whether they want to exit or not $t = T$ is not in the strategy space. The domain $[0, T)$ is straightforward.
The question remains, whether this can be an equilibrium of the whole game. Assume your opponent would always stay in with probability one. Would you always stay in with probability one, too? No - in this case the crash would occur with probability one and your payoff would be $-C$. Hence, a player should leave earlier. But at every $t < T$ we have the problem that each player should stay with probability one. We have a contradiction. This completes the nonexistence proof.■

The nonexistence result is similar to the commitment case. In principle, a player who knows that the opponent will not leave, wants to leave as late as possible. But the problem max t, s.t. $t < T$ does not have a solution.

This Continuous Chicken Game with continuous decision making is formally equal to an ascending all pay auction with punishment. At each point in time players learn whether the opponent exited.

5 Mixed Models

As an application we now try to extend the model. We want to analyze mixed models where $v(t)$ can be positive or negative. In real life the effect of elapsing time on the payoff may be uncertain or vary over time respectively. Consider a Cournot Duopoly with fixed costs where firms have to pay interest on their capital costs. Assume that demand is stochastic. This implies that profits are stochastic. They may be positive or negative. Firms will make positive payoffs if and only if demand is high enough.
We try to capture this by the assumption that $v(t)$ evolves according to a stochastic process. We take the standard Wiener Process.

Assumption 4 : $v(t)$ evolves according to a standard Wiener Process $W(t)$:
(1) $W(0)$ almost surely
(2) $W(t) - W(s) \sim N(0, t-s)$
(3) $W(t-s)$ and $W(v-u)$ $v > u > t > s$ are independent random variables
(4) The paths are continuous
Additionally we assume $C < \infty$, $G < \infty$

We keep the assumption that at time T a new technology is invented that requires at least one firm to exit in order to make positive profits. We could take any stochastic process, but it seems reasonable to choose this special one. First it is widespread in Economics. Second it is continuous and third it is quite simple to handle.

We assume the players to be risk-neutral. The other model assumptions remain the same. Each firm's expected payoff at time $t = 0$ is equal to:

$$EU_{i0}(t_i, t_j) = \begin{cases} \int_0^{t_j} e^{-rw} E_0 v_i(w) dw + e^{-rt_j} \frac{G}{r} & \text{if} \quad t_j < t_i < T \\ \int_0^{t_i} e^{-rw} E_0 v_i(w) dw & \text{if} \quad t_i \leq t_j < T \\ -e^{-rT} C & \text{if} \quad t_i = t_j = T \end{cases}$$

At any time t, firm's expectation about the future $v(t)$ is equal to the value $v(t)$. We can write this as $Ev(t) = v(t)$. Assume we are at a time τ. Firm i's expectation about the future can be represented by figure 3:

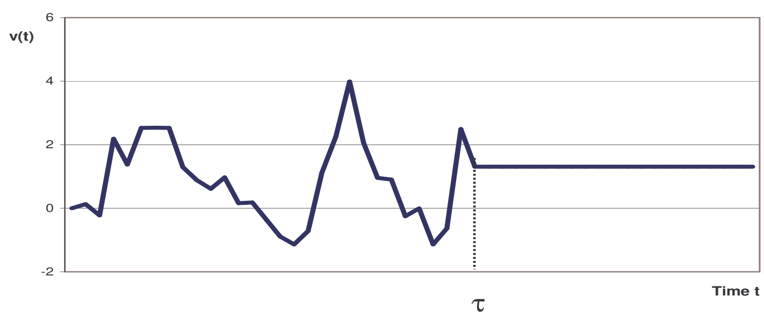

Figure 3: $v(t)$ And Its Expectation Beyond τ

We only analyze the continuous decision case. The set of feasible actions at date $t_0 \in \mathcal{T} = [0...T)$ is $p \in [0...1]$.We are interested in a Nash equilibrium. A strategy is a sequence of exit probabilities that each player announces, given the opponent did not exit yet.

Now each player has to build up expectations about $v(\dot{t})$. But we know $Ev(\dot{t}) = 0$. Because of risk-neutrality players only care about expectations. At any time τ, the firms expect $v(t)$ to be constant. This implies that a firm's problem at a time τ is quite simple.

Now we have to come back to the question whether it is important when the players make their exit decision. The problem here is slightly different from the former models. Assume the players have to choose their exit time at the very beginning and can not change it. In this case the expected value of $v(t)$ is equal to zero for every t. This implies that a firm would choose its exit probability just because of G,$-C$ and r. This implies that we have the ordinary 2x2 Chicken Game in this case. We have to reason that if a firm could not change its exit time at a time t the problem reduces to the very simple 2x2 game.

Lemma 4 : There exists a lower bound B such that any firm exits immediately.

Proof of Lemma 4 : The Wiener process has the property that $W(t)-W(s) \sim N(0, t-s)$. Four our model this is $E\dot{v(t)} = 0$. The expected costs of staying at time t_0 is therefore $v(t_0)$. If this is smaller than the net present value of the "prize" e.g. $\frac{G}{r}$, a firm will exit immediately.

Lemma 5 : The continuous exit game with a time dependence according to a Wiener Process and punishment and risk-neutral players has an equilibrium if $T \to \infty$.

Proof of Lemma 5 : A Wiener process has the following properties: $W(0) = 0$ a.s., $W(t) - W(s) \sim N(0, t-s)\ \forall\ t > s$, any realization $W(t)$ is continuous with probability 1 . This implies that $\Pr[v(t_0) > \frac{G}{r}]$, s.t. $t_0 < T = 1$, if T sufficiently large. If we run the stochastic process long enough, $v(t)$ will take any value because of continuity. Hence, the lower bound will be reached for sure. Lemma 1 implies that both firm will exit the game before T with probability 1.

Theorem 5 : The continuous exit game with $v(t)$ evolving according to a standard Wiener process and punishment does not have an equilibrium.

Proof of Theorem 5 : The previous theorems imply that players should behave as follows:

(1) If $v(t) \geq 0$ stay in with probability 1

(2)If $v(t) < 0$ there are three equilibria; (exit,stay), (stay,exit) or exit with probability according to the mixed strategy equilibrium.
(3) If $v(t) < B$ exit immediately.
As we have a contradiction between point 1 and 2, an equilibrium does not exists. The nonexistence result of the continuous Chicken Game is preserved.

We have to point out that players should behave according to the previous theorems. It may be that they may reach the end of the game (if $v(t)$ remains negative). But if $v(t)$ is positive, we are not able to give them an equilibrium strategy, i.e. a complete contingent plan that is optimal.

6 Conclusion

This paper deals with two basic papers of economic theory: the Chicken Game and the War of Attrition Game.

We provide a simple extension to the Chicken Game and present it in a continuous environment. In a first step we show that this setup is very similar to the War of Attrition Game. We present the general model and show that the continuous Chicken Game and a modified War of Attrition Game are special cases of a more general continuous exit game. We augment the War of Attrition model by a point in time until which at least one firm should have left the market. This assumption may be reasonable because of credit-guidelines or a breakdown of a common.

The models differ in a component that introduces the dependence of the expected payoff from elapsing time. We parameterize this by a simple time dependent variable that enters the expected payoff function. In the War of attrition game this part is negative, in the Chicken Game it is positive.

Despite the model structure is quite simple, the results are fundamentally different. A Nash Equilibrium does not exist for the Chicken Game. The result is important for different reasons. It is a simple game that lacks an equilibrium and it is a problem people face in reality. Two noncooperative opponents go together up to the last straw when the system collapses. If both go beyond this line each of them will suffer serious losses. We provide several examples for the story.

The setup may be explained by pointing out the auction parallels. The analogy to auction theory is obvious. The fact that war of attrition is an auction has been pointed out many years ago (see Milgrom (2004)). Because of the similarity of the two models the auction analogy of the continuous Chicken Game is comprehensible. In the commitment case the War of Attrition Game is equal to a sealed bid second price all-pay auction. The continuous Chicken Game would be a sealed bid second price all-win auction with punishment. We call this class of auction a target auction.

The latter does not exist yet. We agree that an auction where bidders get paid the higher they bid is not economics in the sense of scarce resources, but we provide several example that may apply. Nevertheless the auction analogy helps to understand the setup - especially the difference between a once-and-forever decision (commitment case=sealed bid auction) and continuous decision making case (=English auction). The auction analogy would be very useful if we add uncertainty. But this would be a subject for an additional paper.

Future research may have to solve the following problems. First of all it should fix the nonexistence problem. Further research fields could be the all-win auction and the extension to more general stochastic processes. The natural extension to uncertainty and private information may be supplementary topics.

References

[1] Anton J.J., D.A. Yao: 1992, "Coordination in Split Award Auctions", The Quarterly Journal of Economics, 107(2), 681-707.

[2] Baye M.R., D. Kovenock, C.G. de Vries: 1993, "An Application of the All-Pay Auction", The American Economic Review, 83(1), 289-294.

[3] Bliss, C., B. Nalebuff: 1984, "Journal of Public Economics", Journal of Public Economics, 25, 1-12.

[4] Brams S.J., 1980, "Game-Theoretic Implications of God's Omniscience", Mathematics Magazine, 53(5), 277-282.

[5] Broecker T.: 1990, "Credit-Worthiness Tests and Interbank Competition", Econometrica, 58(2), 429-452.

[6] Bulow J., P.Klemperer, 1999, "The Generalized War of Attrition", The American Economic Review, 89 (1), 175-189.

[7] Dasgupta, P., Maskin E., 1986, "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory" , The Review of Economic Studies, 53, 1-26.

[8] Dasgupta, P., Maskin E., 1986, "Dasgupta P. E. Maskin :The Existence of Equilibrium in Discontinuous Economic Games, II: Applications" , The Review of Economic Studies, 53, 27-41.

[9] Fudenberg, D., J. Tirole, 1991, "Game Theory" , MIT Press, London.

[10] Fudenberg, D., J. Tirole, 1983, "Learning-by-Doing and Market Performance" , The Bell Journal of Economics, 14(2),522-530.

[11] Ghemawat, P., B. Nalebuff: 1985, "Exit" , The RAND Journal of Economics, 16(2),184-194.

[12] Gibbons, R. 1992, "A Primer In Game Theory" , FT Prentice Hall.

[13] Hendricks, K., C. Wilson 1985, "The War of Attrition in Discrete Time" , C.V. Starr Working Paper, R.R. 85-32, 1985.

[14] Hendricks, K., A. Weiss, C. Wilson 1988, "The War of Attrition in Continuous Time with Complete Information" , International Economics Review, 29 (4), 663-680.

[15] Klemperer, P. D. 1999, "Auction theory: A guide to the literature" , Journal of Economic Surveys, 13, 227-286.

[16] Klemperer, P. D., J. Bulow 1996, "Auctions Versus Negotiations" , American Economic Review, 86(1), 180-194.

[17] Krishna, V.: 1997, "An Analysis of the War of Attrition and the All-Pay Auction" , Journal of Economic Theory, 72, 343-362.

[18] Krishna, V., 2002, "Auction Theory" , Academic Press, London.

[19] Maynard Smith, J.: 1974, "The Theory of Games and the Evolution of Animal Conflicts" , Journal of Theoretical Biology, 47, 209-221.

[20] Milgrom,P., J. Roberts: 1982, "Limit Pricing and Entry under Incomplete Information: An Equilibrium Analysis", Econometrica, 50(2), 443-460.

[21] Milgrom,P.,R.J. Webber: 1982, "The Value of Information in a Sealed Bid Auction", Journal of Mathematical Economics, 10, 105-114.

[22] Milgrom, P.: 2004, "Putting Auction Theory to Work", Cambridge University Press, Cambridge UK.

[23] Moldovanu,B., A. Sela: 2001, "The Optimal Allocation of Prizes in Contests", The American Economic Review, 91(3), 542-558.

[24] NALEBUFF, B.J., J.E. STIGLITZ: 1983, "Prizes and Incentives: Towards a General Theory of Compensation and Competition", The Bell Journal of Economics, 14(1), 21-43.

[25] RAPAPORT, A., M. GUYER: 1966, "A Taxonomy of 2x2 Games", General Systems: Yearbook of the Society for General Systems Research, 11, 203-214.

[26] RILEY, J.G.: 1980, "TStrong Evolutionary Equilibrium and the War of Attrition" , Journal of Theoretical Biology, 82, 383-400.

[27] RILEY J.G., W.F. SAMUELSON: 1981, "Optimal Auctions", The American Economic Review, 71(3), 381-392.

[28] SCHARY, A.: 1991, "The Probability of Exit", The RAND Journal of Economics, 22(3), 339-353.

[29] SION, M., P. WOLFE: 1957, "On a Game without a Value", Contributions to the Theory of Games III, Princeton: Annals of Mathematical Studies 39, 299-306.

[30] TIROLE,J., D. FUDENBERG: 1986, "A Theory of Exit in Duopoly", Econometrica, 54(5), 943-960.

[31] WRIGHT B.D.: 1983, "The Economics of Invention Incentives: Patents, Prizes, and Research Contracts", The American Economic Review, 73(4), 691-707.