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Hotelling's Beach with Linear and Quadratic Transportation Costs: Existence of Pure Strategy Equilibria

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Abstract

In Hotelling type models consumers have the same transportation cost function. We deviate from this assumption and introduce two consumer types. Some consumers have linear transportation costs, while the others have quadratic transportation costs. If at most half the consumers have linear transportation costs, a subgame perfect equilibrium in pure strategies exists for all symmetric locations. Furthermore, no general principle of differentiation holds. With two consumer types, the equilibrium pattern ranges from maximum to intermediate differentiation. The degree of product differentiation depends on the fraction of consumer types.

Keywords: Hotelling, Horizontal Product Differentiation, Equilibrium
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1 Introduction

Consumers assess the difference between ideal and actual good differently. For instance, consider consumers whose ideal polo shirt brand is *Lacoste*. If these consumers wear a polo shirt from *Quicksilver*, say, they incur a disutility. This disutility may vary among consumers. Disutility varies because consumers differently assess the distance between ideal and actual good.

The observation that consumers value distance differently allows us to modify Hotelling's model of horizontal product differentiation. We use the modification to address equilibrium existence for Hotelling's model[1]. Our modification consists in introducing two types of consumers. Besides varying tastes, consumers differ in the assessment of the distance between ideal and actual good. Some consumers have linear transportation costs. The other consumers have quadratic transportation costs. This specification represents a hybrid between Hotelling's original formulation and the modification by d'Aspremont, Gabszewicz, and Thisse[2].

Let us motivate again different consumer types using Hotelling's cider example. We can view the firms' locations as the degree of sourness in the cider they offer. Consumers differ in the degree of sourness they desire. Now, consider consumers who prefer the most sour cider possible. All these consumers have the same address. If they consume the sweetest cider possible the distance between their preferred and their consumed good is the same. But these consumers do not necessarily attach the same importance to distance. Consumers value the distance between ideal and consumed good differently.

[1]Existence of equilibrium in Hotelling type models depends on basic assumptions and a number of parameters. Brenner (2001) provides a nice survey about the determinants of equilibrium existence and product differentiation.

[2]We use linear and quadratic transportation cost functions, as these types are well known and widely used in literature.

With our modification we remain very close to Hotelling's model. But we find pure strategy equilibrium existence for any symmetric locations if at most half the consumers have a disutility linear in distance. By contrast, no equilibrium in pure strategies exists for all symmetric locations if more than half the consumers have linear transportation costs.

Previous studies with modifications of Hotelling's model reject a general principle of differentiation[3]. We also reject a general principle of differentiation. With two consumer types, differentiation between firms' goods depends on the fraction of the respective types. However, maximum differentiation is frequent. Firms locate at the extremes in product space for fractions of consumers with linear transportation costs between zero and one third. When the fraction of consumers with linear transportation costs exceeds one third, firms move towards each other. Equilibrium locations are interior solutions. If the number of consumers with linear transportation costs is high (approximately 0.86) the equilibrium distance between firms increases again. This increase is due to restrictions for location spaces that we impose to solve the non-existence problem. Firms must keep a minimal required distance. For large fractions of consumers with linear transportation costs firms locate as close to each other as the minimal required distance allows. The minimal required distance between firms is increasing in the fraction of consumers with linear transportation costs. Hence, product differentiation also increases.

The paper is organized as follows: In section 2 we set up Hotelling's model with two consumer types. Next, in section 3, we derive the demand functions and the equilibrium. In section 4 we discuss the equilibrium outcome. Finally, we conclude in section 5.

[3] See, e.g., Böckem (1994), Economides (1986), Hinloopen and van Marrewijk (1999), and Wang and Yang (1999).

2 The Model

Consider two firms, 1 and 2, each selling one good. The goods are identical except for a one dimensional characteristic. This characteristic represents for example the sweetness of cider or a firm's brand. Firms choose the amount of characteristic by locating on a line with length one. Each firm's location $q_i \in [0, 1]$ measures the amount of characteristic embodied in the good. We assume that firm 1 locates to the left of firm 2, i.e., $q_1 < q_2$. Firm i sells its good at mill price p_i. Let us also assume, for simplicity, that both firms produce at zero fixed and marginal costs.

Suppose there is a continuum of consumers with total mass one. All consumers have the same gross valuation r for exactly one unit of the good. The valuation r is sufficiently high such that in equilibrium all consumers buy from one of the firms. So, valuation r is never binding and the market always covered.

Each consumer knows her individually preferred amount of characteristic embodied in the good. Denote a consumer's most preferred amount of characteristic by the address θ. If a consumer buys a good with a different-than-ideal characteristic, she suffers a disutility. This disutility is the distance between q and θ weighted by the utility loss per unit distance t. Thus, a consumer with address θ pays the mill price p and transportation costs $t|q - \theta|$ when buying a good with characteristic q. We call the mill price plus the transportation costs the generalized price.

Up to this point we follow Hotelling's original model. Our modification consists in modelling two types of consumers. A fraction $\alpha \in [0, 1]$ of consumers incur linear transportation costs. The other fraction $(1 - \alpha)$ of consumers have quadratic transportation costs. We denote a consumer's address who has linear transportation costs by θ_l. Similarly, we denote a consumer's address who has convex transportation costs by θ_c. Addresses for consumers with linear transportation costs are uniformly distributed on $[0, 1]$ with density α. Analogously, addresses for consumers with quadratic transportation costs are uniformly distributed on the unit interval with density $(1 - \alpha)$.

A consumer who buys a good with characteristic q at price p has a utility depending on her address θ and her transportation costs type:

$$u_l(\theta_l, q, p) = r - t|q - \theta_l| - p,$$
$$u_c(\theta_c, q, p) = r - t(q - \theta_c)^2 - p.$$

We study a two-stage price-then-location game. In the first stage firms simultaneously choose locations bearing in mind the subsequent price equilibrium. Given their locations, firms simultaneously set prices in the second stage. To solve the game we use the solution concept of subgame perfect Nash equilibrium. For both stages we look for equilibria in pure strategies.

3 The Equilibrium

3.1 Demand Specification

For the sake of clarity we derive the demand functions before we determine firms' equilibrium behavior. To derive the demand functions we need the addresses of consumers that are indifferent between buying firm 1's good and firm 2's good: the indifferent consumer with linear transportation costs and the indifferent consumer with quadratic transportation costs. For the indifferent consumer with linear transportation costs we draw on Hotelling's analysis. According to Hotelling, the indifferent consumer with linear transportation costs has address $\hat{\theta}_l = [t(q_1 + q_2) + p_2 - p_1]/(2t)$. Consumers with address $\theta_l \leq \hat{\theta}_l$ buy from firm 1.

For the indifferent consumer with quadratic transportation costs we rely on the calculations by d'Aspremont, Gabszewicz, and Thisse. Therefore, the indifferent consumer with quadratic transportation costs has address $\hat{\theta}_c = \left[t(q_2^2 - q_1^2) + p_2 - p_1\right] / (2t(q_2 - q_1))$. All consumers with address $\theta_c \leq \hat{\theta}_c$ shop at firm 1.

Implicitly, we assume that $\hat{\theta}_l$ and $\hat{\theta}_c$ lie between q_1 and q_2. It turns out that this is implied by existence of pure strategy equilibria in the price game.

The distributional assumptions for addresses and indifferent consumers' addresses give the following demand functions D_i for firm i's good:

$$D_1 = \alpha\hat{\theta}_l + (1-\alpha)\hat{\theta}_c, \text{ and } D_2 = \alpha(1-\hat{\theta}_l) + (1-\alpha)(1-\hat{\theta}_c).$$

3.2 The Firms' Equilibrium Behavior

To find the subgame perfect equilibrium we solve the location-then-price game by backwards induction. In the second stage we look for a Bertrand-Nash equilibrium in prices. That is, firm i takes locations and p_j as given and chooses p_i to maximize profits $\pi_i = p_i D_i$. The firms maximization problems are

$$\begin{aligned}
\max_{p_1} \pi_1 = & \max_{p_1} p_1 \left[\alpha(t(q_1+q_2) + p_2 - p_1)\right. \\
& \left. +(1-\alpha)(t(q_2^2 - q_1^2) + p_2 - p_1)/(q_2 - q_1)\right] / (2t), \\
\max_{p_2} \pi_2 = & \max_{p_2} p_2 \left[\alpha(2t - t(q_1+q_2) - p_2 + p_1)\right. \\
& \left. +(1-\alpha)(2t(q_2 - q_1) - t(q_2^2 - q_1^2) - p_2 + p_1)/(q_2 - q_1)\right] / (2t).
\end{aligned}$$

The F.O.Cs. for the firms' maximization problems yield their price reaction functions:

$$p_1(p_2) = p_2/2 + t(q_2^2 - q_1^2)/\left(2(1-\alpha(1-q_2+q_1))\right),$$
$$p_2(p_1) = p_1/2 + t(2(q_2-q_1)+q_1^2-q_2^2)/(2(1-\alpha(1-q_2+q_1))).$$

Note that $\partial^2\pi_i/\partial p_i^2 < 0$ for all α by the assumption $q_1 < q_2$. Both profit functions are strictly concave in prices and the second order conditions are satisfied. It follows that the F.O.Cs. yield the optimal price reaction functions.

The reaction functions are linearly increasing functions of the other firm's price. Therefore, we can solve the system of equations given by the reaction functions to calculate the Bertrand-Nash equilibrium prices. The firms' equilibrium prices in the second stage, given their locations, are

$$p_1^*(q_1,q_2) = t(2+q_1+q_2)(q_2-q_1)/(3(1-\alpha(1-q_2+q_1))), \quad (1)$$
$$p_2^*(q_1,q_2) = t(4-q_1-q_2)(q_2-q_1)/(3(1-\alpha(1-q_2+q_1))). \quad (2)$$

So far we neglected the possibility that firms can sell to consumers in the other firm's hinterland. In Hotelling's original model a firm can lower its price and attract the consumers in the rival's back yard too. D'Aspremont, Gabszewicz, and Thisse show that the firms start undercutting each other's price if they are located too closely. The firms undercut each other if $p_1^*(q_1,q_2)$ and $p_2^*(q_1,q_2)$ are not globally profit-maximizing. Then, the same problem as in Hotelling's original model arises. For $p_1^*(q_1,q_2)$ and $p_2^*(q_1,q_2)$ to constitute Bertrand-Nash equilibrium prices, the firms must not undercut. Following d'Aspremont, Gabszewicz and Thisse, the firms do not undercut each other if

$$p_1^*(q_1,q_2)D_1 \geq (p_2^*(q_1,q_2) - t(q_2-q_1))(\alpha + (1-\alpha)\hat{\theta}_c^1), \tag{3}$$

$$p_2^*(q_1,q_2)D_2 \geq (p_1^*(q_1,q_2) - t(q_2-q_1))(\alpha + (1-\alpha)(1-\hat{\theta}_c^2)), \tag{4}$$

with the demand for good 1 and good 2

$$D_1 = (2+q_1+q_2)/6, \quad D_2 = (4-q_1-q_2)/6.$$

Note that $\hat{\theta}_c^1$ and $\hat{\theta}_c^2$ are the indifferent consumers' addresses for $\overline{p}_i = p_j^*(q_1,q_2) - t(q_2-q_1)$: $\hat{\theta}_c^1 = (q_1+q_2+1)/2$ and $\hat{\theta}_c^2 = (q_1+q_2-1)/2$.

At this point we focus on symmetric locations. Hence, $q_1+q_2 = 1$. It follows that the indifferent consumers with quadratic transportation costs are $\hat{\theta}_c^1 = 1$ and $\hat{\theta}_c^2 = 0$. This means, the undercutting firm serves the entire market by charging $\overline{p}_i$. At the undercutting price $\overline{p}_i$ and with symmetric locations both conditions (3) and (4) simplify to:

$$1/2 \geq (3\alpha + 3\alpha q_1 - 3\alpha q_2)/3.$$

For an equilibrium to exist, the distance between the firms must satisfy $q_2 - q_1 \geq \underline{d}(\alpha) = (2\alpha-1)/(2\alpha)$. We call $\underline{d}(\alpha)$ the minimum required distance.

It is important to discuss the minimum required distance $\underline{d}(\alpha) = (2\alpha - 1)/(2\alpha)$ in more detail. We discuss the minimum required distance for interior locations because $\underline{d}(\alpha)$ is at most 1/2, i.e., $\underline{d} = 1/2$ if $\alpha = 1$. For $\alpha \leq 1/2$ the required distance is never greater than zero. Consequently, firms never find undercutting profitable. Let us give an intuition why undercutting is not profitable for $\alpha \leq 1/2$. With symmetric locations firms' prices are the same, i.e., $p_1^*(q_1,q_2) = p_2^*(q_1,q_2)$. To gain the entire market, firm i reduces its price by $t(q_2 - q_1)$. But this higher demand comes at the expense of a price reduction $t(q_2 - q_1)$ for consumers that already buy from firm i. This expense is high if the price reduction is high relative to the price $p_i^*(q_1, q_2)$. The price reduction is high relative to $p_i^*(q_1, q_2)$ if α is small. For $\alpha \leq 1/2$ a gain in market share does not compensate for the loss due to a lower price. Undercutting is not profitable. However, with an increasing α the price reduction becomes smaller relative to $p_i^*(q_1, q_2)$. Undercutting becomes more attractive. Since the minimum required distance is positive for $\alpha > 1/2$, firms must keep this distance for an equilibrium to exist. The minimum required distance $\underline{d}(\alpha)$ increases with the fraction α of consumers with linear transportation costs. For $\alpha = 1$ we have the polar case Hotelling's original model. Firms must be located outside the quartiles for an equilibrium in pure strategies.

We state the findings from the discussion of the minimum required distance in Lemma 1 and 2.

Lemma 1 *In Hotelling's location-then-price game with two types of consumers and $q_1 + q_2 = 1$ a pure-strategy Bertrand-Nash equilibrium always exists for $\alpha \leq 1/2$. The price equilibrium is given by $p_1^*(q_1, q_2)$ and $p_2^*(q_1, q_2)$.*

Lemma 2 *In Hotelling's location-then-price game with two types of consumers and $q_1+q_2 = 1$ a pure-strategy Bertrand-Nash equilibrium for $\alpha > 1/2$ exists iff $q_2 - q_1 \geq \underline{d}(\alpha)$. If a price equilibrium exists, it is given by $p_1^*(q_1, q_2)$ and $p_2^*(q_1, q_2)$.*

Lemma 2 has a crucial impact on equilibrium existence in the whole two-stage location-then-price game. According to Lemma 2, no price equilibrium in pure strategies exists for $\alpha > 1/2$ and location combinations that violate $q_2 - q_1 \geq \underline{d}(\alpha)$. For these location combinations firms cannot know their payoffs because no price equilibrium exists. Without knowledge of their payoffs, firms do not have the basis for a rational location decision. Therefore, we must restrict firms' strategy spaces for locations in case $\alpha > 1/2$.

The restriction of firms' location spaces is symmetric around the center because we focus on symmetric locations. A symmetric restriction means that firms cannot locate closer to the center than half the minimum required distance. Firm 1 to the left and firm 2 to the right of the center. For firm 1 the restricted strategy space is $q_1 \in [0, (1 - \underline{d}(\alpha))/2]$. By symmetry, firm 2 chooses locations $q_2 \in [(1 + \underline{d}(\alpha))/2, 1]$.

We now turn to the first stage in the location-then-price game. In the first stage, firms simultaneously choose their locations. Firm i maximizes profits π_i with respect to its location q_i. Substituting equilibrium prices for the second stage given by equations 1 and 2 into firms' profit functions, firms' maximization problems are[4]

$$\max_{q_1} \quad t(2 + q_2 + q_1)^2 (q_2 - q_1)/(18(1 - \alpha(1 - q_2 + q_1))),$$

$$\max_{q_2} \quad t(4 - q_1 - q_2)^2 (q_2 - q_1)/(18(1 - \alpha(1 - q_2 + q_1))).$$

Differentiating firms' profits with respect to locations yields the following F.O.Cs.:

$$\frac{\partial \pi_1}{\partial q_1} = \frac{t(2 + q_1 + q_2)}{18(1 - \alpha + \alpha q_2 - \alpha q_1)^2} \underbrace{\left[2\alpha(q_1 - q_2)^2 + (1 - \alpha)(q_2 - 3q_1 - 2)\right]}_{A_1} = 0,$$

$$\frac{\partial \pi_2}{\partial q_2} = \frac{t(4 - q_1 - q_2)}{18(1 - \alpha + \alpha q_2 - \alpha q_1)^2} \underbrace{\left[(1 - \alpha)(4 + q_1 - 3q_2) - 2\alpha(q_1 - q_2)^2\right]}_{A_2} = 0.$$

[4]Note that we do not use equilibrium prices simplified by $q_1 + q_2 = 1$ to compute the equilibrium in locations.

A closer look at the F.O.Cs. shows that the relevant terms for firms' optimal locations are A_1 for firm 1 and A_2 for firm 2. Solving $A_i = 0$ for q_i yields firm i's optimal location as reaction function $q_i(q_j)$ of the other firm j's location. The equation $A_i = 0$ is quadratic in q_i and yields two solutions

$$q_1(q_2) = \left(4\alpha q_2 + 3(1-\alpha) \pm \sqrt{16\alpha q_2(1-\alpha) + 9 - 2\alpha - 7\alpha^2}\right)/(4\alpha),$$
$$q_2(q_1) = \left(4\alpha q_1 - 3(1-\alpha) \pm \sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha q_1(1-\alpha)}\right)/(4\alpha).$$

Easy algebra shows that the first solution for firm 1's location reaction function implies $q_1(q_2) \geq q_2$. Similarly, the second solution for firm 2's optimal location yields $q_2(q_1) \leq q_1$. Hence, the economically meaningful reaction function for firm 1 is the second solution and for firm 2 the first solution. To keep track of, we restate the firms' location reaction functions:

$$q_1(q_2) = \left(4\alpha q_2 + 3(1-\alpha) - \sqrt{16\alpha q_2(1-\alpha) + 9 - 2\alpha - 7\alpha^2}\right)/(4\alpha),$$
$$q_2(q_1) = \left(4\alpha q_1 - 3(1-\alpha) + \sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha q_1(1-\alpha)}\right)/(4\alpha).$$

The intersection of the reaction functions gives a closed form solution for an interior Nash equilibrium in locations (that is, one where $0 < q_1 < q_2 < 1$). To show the existence of an interior Nash equilibrium we need the reaction curves behavior. A detailed discussion of the reaction curves is relegated to the appendix. Here, we use the results in the appendix to determine firms' optimal location choices. For $\alpha > 1/3$ an interior Nash equilibrium in locations exists and is given by the system of equations containing firms' reaction functions. Solving the system of equations for firm 1's location yields two solutions:

$$q_1^* = \left(1 + \alpha \pm \sqrt{(1-\alpha)(5\alpha+1)}\right)/(4\alpha).$$

The solution $q_1 = (1 + \alpha + \sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha)$ is not in the strategy space. In particular,

$$(1 + \alpha + \sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha) > \begin{cases} 1, & \text{for } 0 \leq \alpha \leq 1/2, \\ (1 - \underline{d}(\alpha))/2, & \text{for } 1/2 < \alpha \leq 1. \end{cases}$$

Therefore, we can exclude this first solution. Plugging q_1^* in firm 2's reaction function yields its optimal location:

$$q_2^* = \left(3\alpha - 1 + \sqrt{(1-\alpha)(5\alpha+1)}\right)/(4\alpha).$$

For $\alpha = 1/3$ the firms' reaction functions coincide in the corner $(0, 1)$. Easy calculations show that firm 1 chooses $q_1 = 0$ given $q_2 = 1$. Firm 2's optimal location is $q_2 = 1$ given $q_1 = 0$. Indeed, the location pair $(q_1^* = 0, q_2^* = 1)$ is a Nash equilibrium in locations. Firms choose maximum differentiation.

Last, what is the firms' optimal behavior if $\alpha < 1/3$? We know that $A_1 < 0$ and $A_2 > 0$ for $\alpha < 1/3$. It follows that $\partial\pi_1/\partial q_1 < 0$ for firm 1 and $\partial\pi_2/\partial q_2 > 0$ for firm 2. Consequently, each firm increases its profits by moving away as far as possible from the other. Thus, the principle of maximum differentiation also holds for $\alpha < 1/3$.

Figure 1 shows firms' location choices by the solid lines. The shaded area represents the restriction in location spaces. Firms choose symmetric interior locations around the center for $\alpha > 1/3$, i.e., $q_1^* + q_2^* = 1$. Furthermore, the distance $d^* = (\alpha - 1 + \sqrt{(1-\alpha)(5\alpha+1)})/(2\alpha)$ between firms' equilibrium locations is a decreasing function of α. With a higher α the firms increase

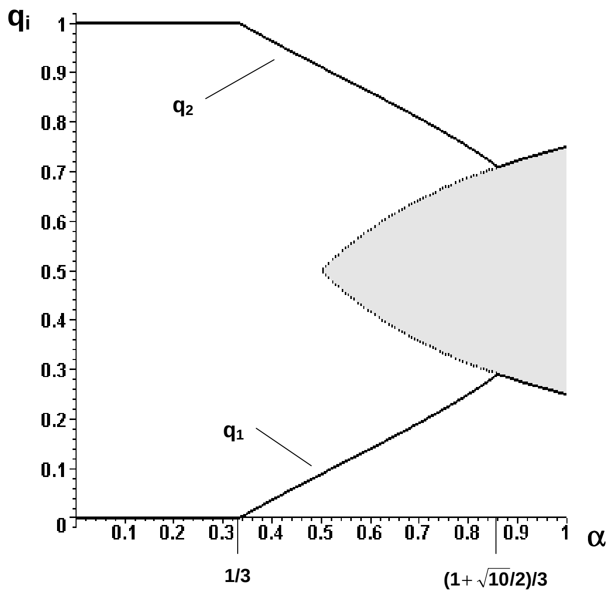

Figure 1: Firms' equilibrium locations

their profits by moving towards each other. But we restrict firms' location spaces for $\alpha > 1/2$. Both firms must maintain half the minimum required distance $\underline{d}(\alpha)$: $q_1^* \leq (1 - \underline{d}(\alpha))/2$ and $q_2^* \geq (1 + \underline{d}(\alpha))/2$. These conditions boil down to

$$0 \leq 6\alpha^2 - 4\alpha - 1$$

for both firms. Now, we see that the condition is satisfied for $1/3 < \alpha < (1 + \sqrt{10}/2)/3$. In this range, firms' optimal locations are given by the solution to the system of equations containing firms' reaction functions. For $\alpha > (1 + \sqrt{10}/2)/3$ firms move as close as possible to the center as strategy spaces allow: $q_1^* = (1-\underline{d}(\alpha))/2$ and $q_2^* = (1+\underline{d}(\alpha))/2$. Because $\underline{d}$ is increasing in α the restriction of location spaces forces firms further apart for $\alpha > (1 + \sqrt{10}/2)/3$.

We summarize the firms' behavior in the location stage with Lemma 3:

Lemma 3 *In Hotelling's location-then-price game with two types of consumers firms choose locations*

$$q_1^* = \begin{cases} 0, & \textit{for } \alpha \leq 1/3, \\ (1+\alpha-\sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha), & \textit{for } 1/3 < \alpha \leq (1+\sqrt{10}/2)/3, \\ (1-\underline{d}(\alpha))/2 = 1/(4\alpha), & \textit{for } (1+\sqrt{10}/2)/3 < \alpha, \end{cases}$$

and

$$q_2^* = \begin{cases} 1, & \textit{for } \alpha \leq 1/3, \\ (3\alpha-1+\sqrt{(1-\alpha)(5\alpha+1)})/(4\alpha), & \textit{for } 1/3 < \alpha \leq (1+\sqrt{10}/2)/3, \\ (1+\underline{d}(\alpha))/2 = 1-1/(4\alpha), & \textit{for } (1+\sqrt{10}/2)/3 < \alpha. \end{cases}$$

We may summarize our findings and describe the equilibrium in Proposition 1.

Proposition 1 *In the Hotelling two-stage location-then-price game with fraction α of consumers with linear transportation costs and fraction $1-\alpha$ of consumers with quadratic transportation costs we find the following equilibria:*

$\cdot$ *if $\alpha \leq 1/3$ (i.e., α is small) firms choose locations $q_1^* = 0$ and $q_2^* = 1$. Firms set the same price $p_1^* = p_2^* = t$ and earn profits $\pi_1^* = \pi_2^* = t/2$,*

$\cdot$ *if $1/3 < \alpha \leq (1+\sqrt{10}/2)/3$ (i.e., α is intermediate) firms choose locations given by Lemma 3. Firms set the same price $p_1^* = p_2^* = p^* = t(1-\alpha-\sqrt{(5\alpha+1)(1-\alpha)})/(\alpha(\alpha-1-\sqrt{(5\alpha+1)(1-\alpha)}))$ and earn profits $\pi_1^* = \pi_2^* = p^*/2$,*

$\cdot$ *if $(1+\sqrt{10}/2)/3 < \alpha$ (i.e., α is large) firms must keep the minimum required distance $\underline{d}(\alpha)$. Firms choose locations $q_1^* = 1/(4\alpha)$ and $q_2^* = 1-1/(4\alpha)$, set the same price $p_1^* = p_2^* = t(2\alpha-1)/\alpha$, and earn profits $\pi_1^* = \pi_2^* = t(2\alpha-1)/(2\alpha)$.*

4 Discussion

We begin the discussion with the degree of price competition. Our specification for the degree of price competition refers to the cross-price sensitivity of demand. The cross-price sensitivity is the amount of consumers firm i gains or loses as firm j changes its price[5]. Thus, the cross-price sensitivity η in our model is:

$$\eta = \partial D_i/\partial p_j = -\partial D_i/\partial p_i = (1 - \alpha(1-d))/(2dt), \quad i = 1, 2,$$

where $d = q_2 - q_1$. With this definition, a higher η indicates more intense price competition. Furthermore, note that η does not account for undercutting effects. But undercutting is ruled out. Therefore, we proceed the discussion about η without considering an undercutting process.

The degree of price competition η depends on parameters t and α as well as on distance d. First, price competition intensifies if t decreases, ceteris paribus. This is characteristic for Hotelling-type models, since t represents consumers' sensitivity to product differentiation. Consumers attach less importance to product differentiation when t is low. When t approaches zero, the model approaches Bertrand competition with homogeneous goods.

Secondly, the degree of price competition is decreasing in α, given t and d: $\partial\eta/\partial\alpha = -(1-d)/(2dt) \leq 0$. Price competition becomes less intense, the higher the fraction of consumers with linear transportation costs. Consumers with linear transportation costs, l-consumers, compare the utility from consuming good 1 with the utility from good 2. So do consumers with quadratic transportation costs, the c-consumers. This utility comparison reduces to a

[5] Brenner (2001) uses the cross-price sensitivity of demand as a measure for the degree of price competition to highlight the relationship between transportation cost functions and equilibrium existence.

comparison of the difference in transportation costs with the price difference. We know that l-consumers have a greater difference in transportation costs than c-consumers. Thus l-consumers care less for a price change than c-consumers. A price change has a weaker effect on l-consumers. If α increases more consumers care less for the price relative to travel distance. The degree of price competition decreases.

Note that a comparison for competition intensity between Hotelling's original model and the modified version of d'Aspremont, Gabszewicz, and Thisse confirm the observation $\partial\eta/\partial\alpha < 0$. In the polar case Hotelling the degree of price competition is less intense than in the other polar case d'Aspremont, Gabszewicz, and Thisse.

The third factor that affects the degree of price competition is the distance $d = q_2 - q_1$. Keeping t and α constant, the degree of competition is decreasing in d, i.e., $\partial\eta/\partial d = (-1+\alpha)/(2d^2 t) \leq 0$. By moving towards each other the firms offer less differentiated goods. For consumers, less differentiation leads to better substitutability between goods. Price competition increases.

Figure 2 illustrates the equilibrium degree of price competition η^* for various t by solid lines. The dotted line is the equilibrium distance d^* between firms.

In the range $\alpha \leq 1/3$ firms choose maximum differentiation. The degree of price competition is constant. For $1/3 < \alpha \leq (1+\sqrt{10}/2)/3$ both firms move towards the center. The distance d^* and product differentiation decrease. Because products are less differentiated price competition is more intense. As soon as $\alpha > (1+\sqrt{10}/2)/3$ the degree of price competition decreases. Two effects that work in the same direction relax price competition. With an increasing α we move closer to Hotelling's original model. As argued above the degree of price competition is lower the closer

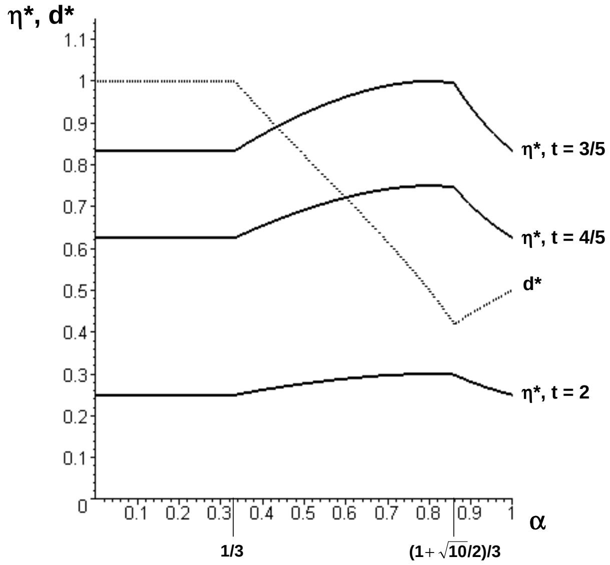

Figure 2: The degree of competition for various t and the equilibrium distance d^* between firms

we are to Hotelling's original model. The second effect stems from the restriction of location spaces. For $\alpha > (1 + \sqrt{10}/2)/3$ firms move away from each other because they must keep the minimum required distance $\underline{d}$. Because $\partial \underline{d}(\alpha)/\partial\alpha > 0$ the distance between firms increases. Firms offer more differentiated goods. More differentiated goods soften price competition.

Let us now discuss firms' location choices in the second stage. Proposition 1 and Figure 1 show that no general principle of differentiation exists in our model. Differentiation depends on the fraction of l- and c-consumers in the way intuitively expected. The more consumers with linear transportation costs, the closer we are to Hotelling's model and the closer firms move to each other. However, maximum differentiation is frequent for the range $\alpha \leq 1/3$. It seems that maximum differentiation is quite robust.

Two now standard opposite effects are responsible for firms' location choices. On the one hand, firms differentiate their goods to weaken price competition. This is the price effect. Because a larger distance between firms reduces the degree of price competition firms want to move away from each other. On the other hand, firms move inwards in the product space to capture a larger market share. This centripetal force is the demand effect. The relative strength of those effects determines the location pattern in equilibrium.

The price effect dominates the demand effect if the fraction of consumers with linear transportation costs is small. In this case, the principle of maximum differentiation holds. By contrast, maximum differentiation is not the equilibrium outcome for intermediate and large α. The reason is that the demand effect does not depend on α while the price effect does. With an increasing α the degree of price competition decreases. Relative to the demand effect the price effect becomes weaker. The price effect does not overcompensate the demand effect anymore. Firms balance the trade-off between price and demand effect increasingly in favor of the latter. Since the trade-off is increasingly in favor of the demand effect, firms move towards each other for intermediate α. For large α, the demand effect still becomes stronger. But again, the restricted location spaces lead to increased product differentiation.

Last, we turn to the condition that ensures an equilibrium. More precisely, what is the maximum fraction of consumers with linear transportation costs such that an equilibrium in pure strategies exists for all symmetric locations. The answer is short and given by Lemmas 1 and 2: $\alpha \leq 1/2$. At most half the consumers can have linear transportation costs. Otherwise, no pure-strategy price equilibrium exists in the second stage for all symmetric locations. Without price equilibrium for some location patterns firms are not able to evaluate their profits in the first stage. No (pure-strategy) equilibrium to the two-stage location-then-price game exists.

5 Conclusions

Consumers may assess deviations from buying a less-than-ideal good differently. To allow for such different assessment we introduce two consumer types in Hotelling's model of product differentiation. A fraction α of consumers have linear transportation costs. The other fraction $(1 - \alpha)$ of consumers have quadratic transportation costs.

As expected, we cannot support a general principle of differentiation. But maximum differentiation seems to be quite robust. In the subgame perfect Nash equilibrium firms choose maximum differentiation if at most one third of the consumers have linear transportation costs. With an increasing fraction of consumers who have a disutility linear in distance the agglomeration force becomes stronger. Firms move closer to each other.

The fraction of consumers with linear transportation costs also affects equilibrium existence. Only if at most half the consumers have linear transportation costs an equilibrium in the price subgame exists. A price equilibrium no longer exists for any symmetric locations if more than half the consumers have linear transportation costs.

To circumvent the non-existence problem we impose location restrictions on firms. Firms must keep the minimal required distance such that a pure-strategy price equilibrium exists. This minimal required distance must go from zero to one half.

Appendix

Maximizing firms' profits with respect to their locations yields the reaction functions:

$$
\begin{aligned}
q_1(q_2) &= \left(4\alpha q_2 + 3(1-\alpha) - \sqrt{16\alpha(1-\alpha)q_2 + 9 - 2\alpha - 7\alpha^2}\right)/(4\alpha),\\
q_2(q_1) &= \left(4\alpha q_1 - 3(1-\alpha) + \sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha(1-\alpha)q_1}\right)/(4\alpha).
\end{aligned}
$$

Denote the term in the square root in firm i's reaction function by φ_i. Simple inspection of φ_i shows that it is non-negative. For φ_1 in firm 1's reaction function this is

$$\varphi_1 = 16\alpha(1-\alpha)q_2 + \underbrace{9 - 2\alpha - 7\alpha^2}_{\geq 0} \geq 0, \quad \forall \alpha \in [0,1].$$

To see that φ_2 in firm 2's reaction function is also non-negative we first observe that it negatively depends on q_1. Hence, if φ_2 is non-negative for $q_1 = 1$, non-negativity holds for all $q_1 \leq 1$. The problem boils down to checking if φ_2 is non-negative for $q_1 = 1$. Indeed, for $q_1 = 1$, φ_2 is not smaller than zero:

$$\varphi_2 = 9 + 14\alpha - 23\alpha^2 - 16\alpha + 16\alpha^2 \geq 0, \quad \forall \alpha \in [0,1].$$

Both reaction functions are positively sloped:

$$
\begin{aligned}
\frac{\partial q_1(q_2)}{\partial q_2} =& 1 - \frac{2(1-\alpha)}{\sqrt{16\alpha(1-\alpha)q_2 + 9 - 2\alpha - 7\alpha^2}} > 0,\\
\frac{\partial q_2(q_1)}{\partial q_1} =& 1 - \frac{2(1-\alpha)}{\sqrt{9 + 14\alpha - 23\alpha^2 - 16\alpha(1-\alpha)q_1}} > 0.
\end{aligned}
$$

Moreover, the first derivatives show that the slopes are never greater than one. Let us also compare these slopes:

$$
\begin{aligned}
\partial q_1(q_2)/\partial q_2 &\lesseqqgtr \partial q_2(q_1)/\partial q_1 \\
1-\frac{2(1-\alpha)}{\sqrt{16\alpha(1-\alpha)q_2+9-2\alpha-7\alpha^2}} &\lesseqqgtr 1-\frac{2(1-\alpha)}{\sqrt{9+14\alpha-23\alpha^2-16\alpha(1-\alpha)q_1}} \\
16\alpha(1-\alpha)q_2+9-2\alpha-7\alpha^2 &\lesseqqgtr 9+14\alpha-23\alpha^2-16\alpha(1-\alpha)q_1 \\
q_1+q_2 &\lesseqqgtr 1.
\end{aligned}
$$

The comparison of the slopes shows that firm 1's reaction function is less steeper for $q_1 + q_2 < 1$. For symmetric locations, that is $q_1 + q_2 = 1$, firms' reaction functions exhibit the same slope. If $q_1 + q_2 > 1$ firm 1's reaction function is steeper than firm 2's reaction function.

Because the reaction functions are non-linear we need the second derivatives to make further conclusions about their behavior:

$$
\begin{aligned}
\frac{\partial q_1^2(q_2)}{\partial^2 q_2} =& 16\alpha(1-\alpha)^2\varphi_1^{-3/2} \geq 0, \\
\frac{\partial q_2^2(q_1)}{\partial^2 q_1} =& -16\alpha(1-\alpha)^2\varphi_2^{-3/2} \leq 0.
\end{aligned}
$$

Therefore, firm 1's reaction function is strictly convex in q_2. Firm 2's reaction function is strictly concave in q_1.

Next, we evaluate the functions' values at the endpoints of the strategy space.

$$
\begin{aligned}
q_1(q_2=0) &= \left(3(1-\alpha)-\sqrt{9-2\alpha-7\alpha^2}\right)/(4\alpha) \leq 0, \\
q_1(q_2=1) &= \left(3+\alpha-\sqrt{9+14\alpha-23\alpha^2}\right)/(4\alpha) \leq 1, \\
q_2(q_1=0) &= \left(-3(1-\alpha)+\sqrt{9+14\alpha-23\alpha^2}\right)/(4\alpha) \geq 0, \\
q_2(q_1=1) &= \left(7\alpha-3+\sqrt{9-2\alpha-7\alpha^2}\right)/(4\alpha) \geq 1.
\end{aligned}
$$

Finally, we compare $q_2(q_1(q_2 = 1))$ with 1 and $q_1(q_2(q_1 = 0))$ with 0. Firm 2's reaction function evaluated at $q_1(q_2 = 1)$ is:

$$q_2(q_1(q_2 = 1)) = \Big[4\alpha - \sqrt{(23\alpha + 9)(1 - \alpha)} \\ + \sqrt{(1 - \alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1 - \alpha)})}\Big] /(4\alpha).$$

The comparison shows that $q_2(q_1(q_2 = 1)) \lesseqqgtr 1$ dependent on α:

$$\Big[4\alpha - \sqrt{(23\alpha + 9)(1 - \alpha)} + \sqrt{(1 - \alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1 - \alpha)})}\Big] /(4\alpha) \lesseqqgtr 1$$
$$\sqrt{(1 - \alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1 - \alpha)})} \lesseqqgtr \sqrt{(23\alpha + 9)(1 - \alpha)}$$
$$\sqrt{(23\alpha + 9)(1 - \alpha)} \lesseqqgtr \alpha + 3$$
$$\alpha(1 - 3\alpha) \lesseqqgtr 0.$$

For $\alpha < 1/3$ firm 2's reaction function yields a value greater than 1 evaluated at $(q_1(q_2 = 1))$. If $\alpha = 1/3$ and $(q_1(q_2 = 1))$ firm 2's optimal location is 1. For $\alpha > 1/3$ firm 2's reaction function takes a value less than 1 evaluated at $(q_1(q_2 = 1))$.

Similarly, $q_1(q_2(q_1 = 0)) \lesseqqgtr 0$:

$$(\sqrt{(23\alpha + 9)(1 - \alpha)} - \sqrt{(1 - \alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1 - \alpha)})})/(4\alpha) \lesseqqgtr 0$$
$$\sqrt{(23\alpha + 9)(1 - \alpha)} \lesseqqgtr \sqrt{(1 - \alpha)(19\alpha - 3 + 4\sqrt{(23\alpha + 9)(1 - \alpha)})}$$
$$0 \lesseqqgtr \alpha(1 - 3\alpha).$$

For $\alpha < 1/3$ firm 1's optimal location is smaller than 0 evaluated at $q_2(q_1 = 0)$. For $\alpha = 1/3$ firm 1's reaction function takes the value 0. If $\alpha > 1/3$ and $q_2(q_1 = 0)$ firm 1's optimal location is $q_1 > 0$.

References

[1] Anderson, P. Simon, 1979, Equilibrium Existence in the Linear Model of Spatial Competition, Economica, 55(220), pages 479-491.

[2] d'Aspremont, Claude, Gabszewicz, J. Jean, Thisse, Jacques-François, 1979, On Hotelling's "Stability in Competition", Econometrica, 47(5), pages 1145-1150.

[3] Böckem, Sabine, 1994, A Generalized Model of Horizontal Product Differentiation, The Journal of Industrial Economics, 42(3), pages 287-298.

[4] Brenner, Steffen, 2001, Determinants of Product Differentiation: A Survey, Working Paper, Humboldt-University of Berlin.

[5] Economides, Nicholas, 1984, The Principle of Minimum Differentiation Revisited, European Economic Review, 24(3), pages 345-368.

[6] Economides, Nicholas, 1986, Minimal and Maximal Product Differentiation in Hotelling's Duopoly, Economics Letters, 21(1), pages 67-71.

[7] Hinloopen, Jeroen, van Marrewijk, Charles, 1999, On the Limits and Possibilities of the Principle of Minimum Differentiation, International Journal of Industrial Organization, 17(5), pages 735-750.

[8] Hotelling, Harold, 1929, Stability in Competition, The Economic Journal, 39, pages 41-57

[9] Wang, X. Henry, Yang, Z. Bill 1999, On Hotelling's Location Model with a Restricted Reservation Price, Australian Economic Papers, 38(3), pages 259-275.